NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2
Higher 2

NAME

CLASS 2ma2

REGISTRATION NUMBER

MATHEMATICS 9758/01

Preliminary Examination **09 September 2024**

Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, class and registration number in the boxes above.
Please write clearly and use capital letters.

Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing and/or scientific calculator is expected, where appropriate.
All relevant working, statements and reasons must be shown in order to obtain full credit for your solution.

You are reminded of the need for clear presentation in your answers.
Up to 2 marks may be deducted for improper presentation.

The number of marks is given in the brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question Number	Marks Possible	Marks Obtained
1	4	
2	4	
3	5	
4	7	
5	8	
6	8	
7	8	
8	9	
9	10	
10	11	
11	12	
12	14	
Presentation Deduction		– 1 / – 2
TOTAL	100	

This document consists of **7** printed pages.

1 A circular sector has radius r cm and angle θ radians. This sector has area A cm^2 and fixed perimeter k cm.

(i) Show that $\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{k}{2} - 2r$. [2]

(ii) Given that r is increasing at a constant rate of $\frac{k}{10}$ cm s^{-1}, find in terms of k, the rate at which A is changing when the arc length of the sector is equal to the radius. [2]

2 Two of the roots of the equation $z^3 + az^2 + bz + c = 0$ are $3\mathrm{e}^{\mathrm{i}\left(-\frac{2}{3}\pi\right)}$ and -2. Given further that a, b and c are integer constants, find the values of a, b and c. [4]

3 **Do not use a calculator to solve this question.**

(i) Solve the inequality $\frac{x-6}{4x^2+x-5} \geq 1$. [3]

(ii) Hence solve the inequality $\frac{x-6x^2}{4+x-5x^2} \geq 1$. [2]

4 Relative to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors that are not parallel to one another. The points A, B and C are not collinear.

A point of trisection is a point that divides a line segment internally in the ratio $1:2$ or $2:1$. Suppose another two points D and E are points of trisection of line segments AB and AC respectively and both points are nearer to A than to B and C respectively. The lines BE and CD meet at point F.

(i) Show that the vector equations of the lines BE and CD can be expressed as $\mathbf{r} = \frac{2}{3}\lambda\mathbf{a} + (1-\lambda)\mathbf{b} + \frac{1}{3}\lambda\mathbf{c}$ and $\mathbf{r} = \frac{2}{3}\mu\mathbf{a} + \frac{1}{3}\mu\mathbf{b} + (1-\mu)\mathbf{c}$ respectively, where λ and μ are parameters. Hence, show that at point F, $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, where α, β and γ are constants, each to be expressed in terms of λ and μ. [4]

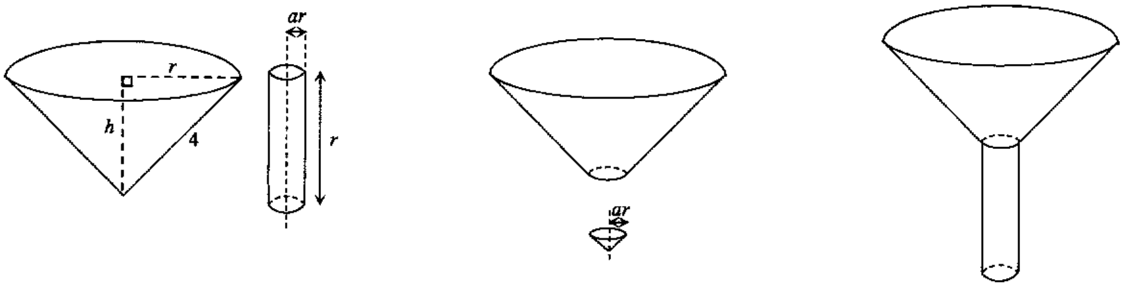
(ii) Given further that $OACB$ is a parallelogram, find the position vector of F in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

5 [The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

A manufacturer makes a funnel-shaped ornament from the same material which consists of two parts as shown in **Figure 1**.

- a right cone of radius r cm, height h cm and a slant height of 4 cm,
- a cylinder with radius ar cm and height r cm, where $0 < a < 1$.

Figure 1 **Figure 2** **Figure 3**



From the original cone, a similar cone with radius ar cm is removed from the vertex as shown in **Figure 2**. The remaining part of the cone is joined to the cylinder to form the funnel as shown in **Figure 3**. It may be assumed that the thickness of the funnel is negligible.

Given that the volume of the ornament is V cm^3, find V in terms of a and r. [3]

For the remainder of this question, assume that $a = 0.25$.

(a) The manufacturer wants V to be a maximum. If $r = r_1$ gives the maximum value of V, show that r_1 satisfies the equation $457r^4 - 9664r^2 + 50176 = 0$. [3]

(b) Show that one of the positive roots to the equation in part **(a)** does not give a stationary value of V. Hence find the value of h for which V is stationary. [2]

6 **(i)** Show that $\frac{e^{i\theta}}{1-e^{i\theta}}$ can be expressed as $k\left(i\cot\frac{\theta}{2} - 1\right)$, where k is a real constant to be determined exactly. [3]

(ii) Express the complex number i in three equivalent $re^{i\theta}$ forms, where $r > 0$ and $-3\pi < \theta \le 3\pi$. [2]

(iii) Hence find the roots of the equation $\left(\frac{w}{w+1}\right)^3 - i = 0$, leaving your answers in the form $k(i\cot\phi - 1)$, where $-\frac{\pi}{2} < \phi \le \frac{\pi}{2}$. [3]

7 (i) Given that $y = \operatorname{cosec}\left(2x + \frac{\pi}{4}\right)$, show that $\frac{d^2y}{dx^2} = 8y^3 - 4y$. [3]

(ii) By further differentiation of the result in part (i), find the first four terms of the Maclaurin series for $\operatorname{cosec}\left(2x + \frac{\pi}{4}\right)$ exactly. [3]

(iii) Hence estimate the value of $\operatorname{cosec}\left(\frac{13\pi}{50}\right)\cot\left(\frac{13\pi}{50}\right)$, giving your answer in the form $\sqrt{2}\left(p + q\pi + r\pi^2\right)$, where p, q and r are rational constants to be determined. [2]

8 (a) The sum, S_n, of the first n terms of a sequence of numbers $u_1, u_2, u_3, \ldots,$ is given by

$$S_n = An^2 + Bn + 2^{n+1},$$

where A and B are non-zero constants. It is also given that the third term is 21 and the fifth term is 53. Find a simplified expression for u_n in terms of n. [4]

(b) (i) Use the method of differences to show that $\sum_{r=1}^{n} \ln\left(\frac{r(r+2)}{(r+1)^2}\right) = \ln\left(\frac{n+2}{n+1}\right) - \ln 2$. [3]

(ii) Hence, find the exact value of $\sum_{r=0}^{n} \ln\left(\frac{r^2+4r+3}{(r+2)^2}\right)$ in terms of n. [2]

9 (a) Show that the curve with equation $y = \left(x^2 + cx\right)e^{-x}$ has two stationary points for all real values of c. [3]

(b) The curves C_1 and C_2 have equations $x^2 + 4y^2 - 6x - 7 = 0$ and $y = \frac{2x-3}{x-1}$ respectively. Write the equation of C_1 in the form $\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$. Sketch, on the same diagram, both C_1 and C_2, indicating clearly their key features as well as the coordinates of their points of intersection. [7]

10 The function f is defined by

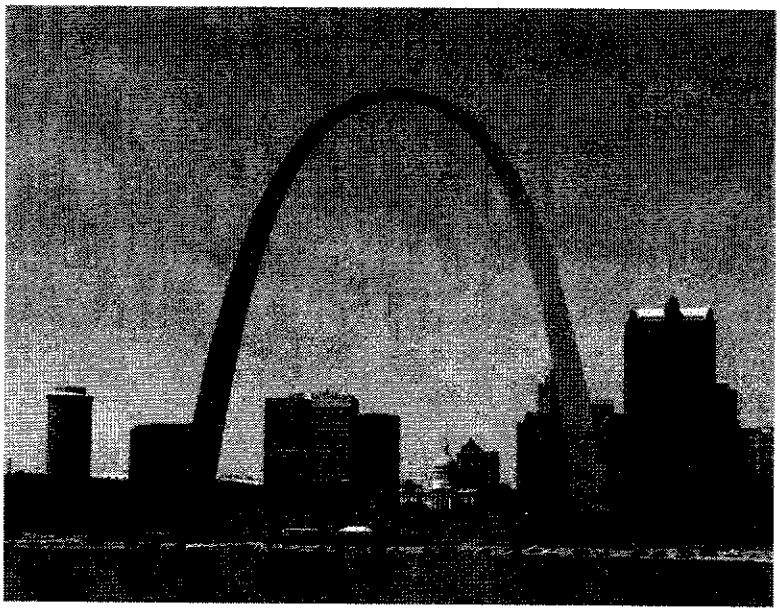
$$f(x) = (x+1)|x+1|, \quad \text{for } x \in \mathbb{R}, -4 < x \le 2.$$

(i) Find f^{-1}. [4]

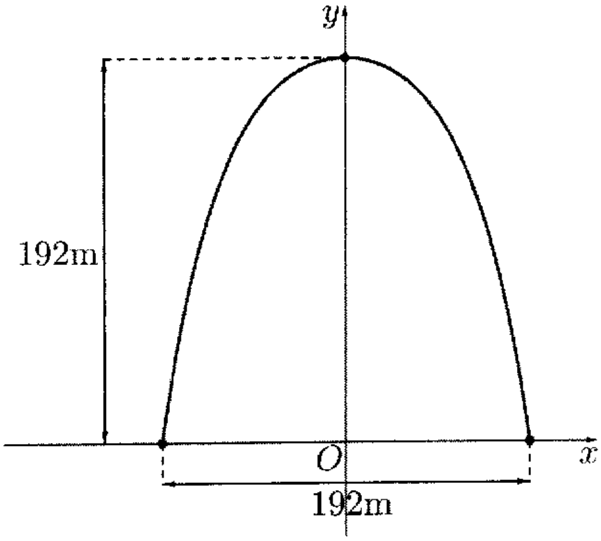
(ii) On the same diagram, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = ff^{-1}(x)$, labelling clearly the coordinates of the end-points. [4]

(iii) Solve exactly the inequality $f(x) \le f^{-1}(x)$. [3]

11 The diagram below shows the Gateway Arch, which is a monument in St. Louis, Missouri, United States. The arch stands at 192 metres tall and is 192 metres wide.



The arch can be modelled by part of a curve as shown in the diagram below.



The highest point of the curve lies on the y-axis and the curve is symmetrical about the y-axis. The two endpoints both lie on the x-axis. It is known that the curve satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = ak\sqrt{1+\left(\frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

for some constants a and k.

(i) Show that the substitution $p = \frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}$ reduces the differential equation to

$$\frac{\mathrm{d}p}{\mathrm{d}x} = a\sqrt{1+p^2}.$$ [1]

(ii) By using the substitution $p = \tan u$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$, to solve the reduced differential equation in part **(i)**, show that

$$p = \frac{\mathrm{e}^{ax} - \mathrm{e}^{-ax}}{2}.$$ [8]

(iii) Given that $a = -0.0329$ and $k = 0.701$, find y in terms of x. [3]

12 The planes π_1 and π_2, which meet in the line l_1, have equations

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 25 \text{ and } \pi_2 : x + ky - 2z = -15,$$

where k is a constant.

Another line l_2 has equation $\mathbf{r} = \begin{pmatrix} -15 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\beta \in \mathbb{R}$.

(i) Determine the position vector of the point on l_1 such that the coordinates of this point is independent of k. Hence find a vector equation of l_1. [4]

(ii) Determine the possible value(s) of k such that l_1 and l_2 are skew. [3]

Assume that $k = 4$ for the rest of this question.

(iii) Points A and B are on l_1 and l_2 respectively such that $\overrightarrow{AB}$ is perpendicular to both lines. Show that $\left|\overrightarrow{AB}\right| = \sqrt{\frac{p}{2}}$, where p is an integer to be determined. [3]

(iv) Find exactly the sine of the acute angle between l_2 and π_1. [2]

It is given further that l_2 lies on a third plane that is perpendicular to l_1, and l_2 intersects π_1 at point P.

(v) Deduce the shortest distance from P to l_1. [2]

NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2
Higher 2

NAME

CLASS 2ma2

REGISTRATION NUMBER

MATHEMATICS 9758/02

Preliminary Examination **16 September 2024**

Paper 2 **3 hours**

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Presentation Deduction		– 1 / – 2
TOTAL	100	

Section A: Pure Mathematics [40 marks]

1 The complex numbers z and w satisfy the following equations.

$$w^* = z - 2\mathrm{i}$$

$$wz^* = |w|^2 + 6\mathrm{i}$$

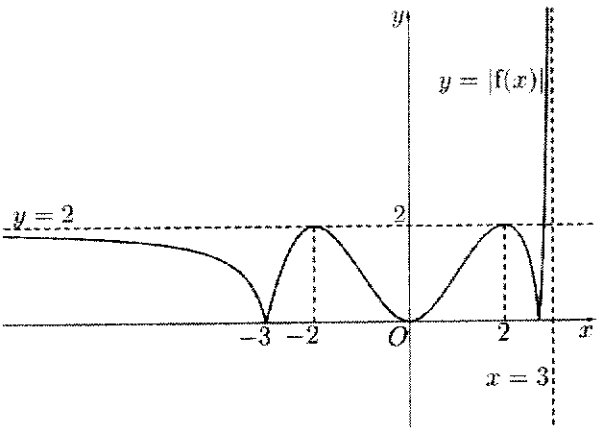
Find z and w, giving your answers in the form $a + \mathrm{i}b$ where a and b are real numbers. [4]

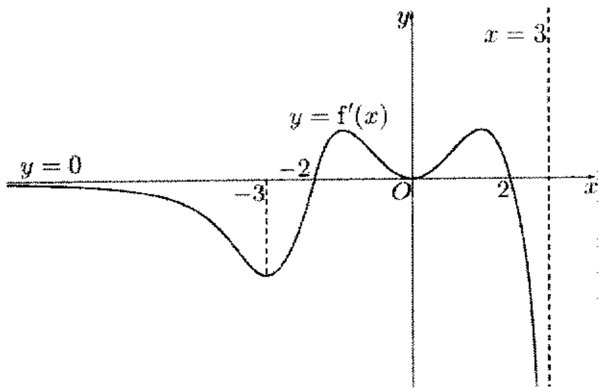
2 An arithmetic series has first term a and common difference d, where a and d are non-zero. A convergent geometric series has first term b and common ratio r, where b is positive and r is non-zero. It is given that the fourth and ninth terms of the arithmetic series are equal to the sixth and ninth terms of the geometric series respectively and the eleventh term of the arithmetic series is br^6 less than the fourteenth term of the geometric series.

(i) Show that r satisfies the equation $5r^8 - 7r^3 - 5r + 2 = 0$ and solve this equation, giving your answer correct to 4 decimal places. [4]

(ii) Using this value of r, deduce that for any positive integer n, the sum of the terms of the geometric series after, but not including, the nth term is less than $\frac{3}{5}b$. [2]

3 The graphs of $y = |\mathrm{f}(x)|$ and $y = \mathrm{f}'(x)$ are given below respectively.





On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(x)$, [3]

(ii) $y = |\mathrm{f}(2 - x)|$, [3]

(iii) $y = \dfrac{1}{\mathrm{f}'(x)}$, [3]

indicating clearly the equations of asymptotes, turning points, axial intercepts and end-points, where applicable and possible.

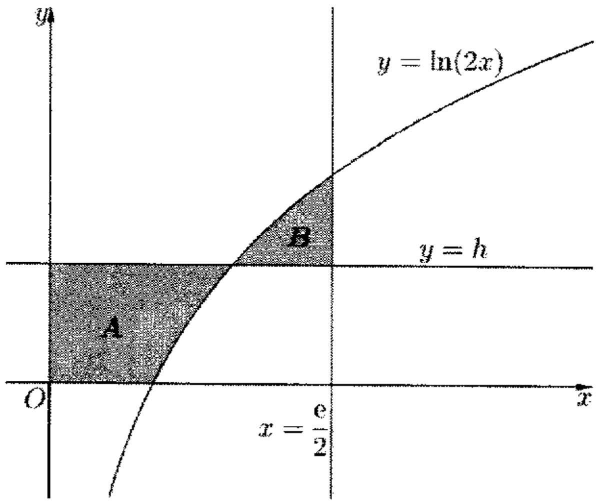
4 **(a)** It is given that $f(x)=\begin{cases}\sin\left(\dfrac{x}{2}\right), & \text{for } 0\le x\le 3\pi,\\ \dfrac{x}{\pi}-4, & \text{for } 3\pi<x<4\pi\end{cases}$

and that $f(x)=f(x+4\pi)$ for all real values of x.

(i) Sketch the graph of $y=f(x)$ for $-\dfrac{\pi}{2}\le x\le 6\pi$. [3]

(ii) Find $\int_{-\frac{\pi}{2}}^{6\pi}|f(x)|\,dx$, leaving your answer in exact form. [2]

(b)



From the diagram above, the region A is bounded by the curve $y=\ln(2x)$, the line $y=h$, $h\in\mathbb{R}$, the x-axis and the y-axis while the region B is bounded by the curve $y=\ln(2x)$ and the lines $x=\dfrac{e}{2}$ and $y=h$. Given that the volumes of the solids generated when A and B are rotated completely about the y-axis are equal, find the exact value of h. [5]

5 In this question, it is given that a is a positive constant. Leave all answers in terms of a where necessary.

(a) Find $\displaystyle\int\frac{x}{\left(1+ax^2\right)^2}\,dx$. Hence find $\displaystyle\int\frac{ax^2}{\left(1+ax^2\right)^2}\,dx$. [5]

(b) Use the substitution $x=\dfrac{1}{y}$ to find the exact value of $\displaystyle\int_{\sqrt{2}a}^{2a}\frac{1}{x\sqrt{x^2-a^2}}\,dx$. [6]

Section B: Probability and Statistics [60 marks]

6 For a positive integer n, it is given that $\mathrm{P}(A=n)=0.009542$ and $\mathrm{P}(A=n+1)=0.004090$, where $A \sim \mathrm{B}(2n+1, p)$. Show that p satisfies an equation of the form $\frac{p}{1-p}=k$, where k is a constant to be determined. Hence find the value of n and the variance of A. [4]

7 A random variable X has mean μ and variance 36.

A random sample of n independent observations of X is taken and the sample mean is denoted by $\overline{X}$. Find the least value of n such that $\mathrm{P}\left(\left|\overline{X}-\mu\right|<0.5\right)>0.98$, stating any assumptions needed at the start of your calculations. [4]

8 At a funfair game stall, players are allowed to choose two cards at random from six cards, with each card labelled with one letter from A to F. The player's score, denoted by X, is the Manhattan distance between the two squares corresponding to the player's two chosen letters on the grid below,

A	B	C
D	E	F

where the Manhattan distance between two squares is the minimum total number of horizontal and vertical steps required to travel between them. For example, the Manhattan distance between B and F is 2, while the Manhattan distance between D and C is 3.

(i) Tabulate the probability distribution of X. [2]

The stall owner charges \$10 per game and rewards the player with a cash prize, in dollars, of $\frac{k}{10}$ times of the square of the player's score, where k is a positive integer.

(ii) Determine the largest value of k for the stall to be profitable in the long run. [4]

9 A factory manufactures cans and bottles of iced tea. Machine A is used to fill the cans with iced tea and Machine B is used to fill the bottles with iced tea. Machine A is set to fill each can with 300 millilitres (ml) of iced tea. A random sample of 60 filled cans of iced tea was taken and the volume, x ml, of iced tea in each can was measured. The following summarised data was obtained.

$$\sum(x-300)=-112.8,\ \sum(x-300)^2=4532.87$$

(a) Defining clearly any symbols you use, test at the 8% level of significance, whether the mean volume of iced tea per can is 300 ml. [6]

(b) Explain in the context of the question, the meaning of 'at the 8% level of significance'. [1]

(c) The manager of the factory claims that the mean volume of iced tea that Machine B fills per bottle with is at least 500 ml. It is found that the volumes of iced tea in the bottles filled by Machine B follow a normal distribution with standard deviation 5 ml. A random sample of 35 filled bottles was taken and a test for the validity of the manager's claim was carried out at the 4% level of significance. Find the critical region for this test, correct to 1 decimal place. [3]

10 Webflix is a video streaming service. The numbers of Webflix subscribers worldwide, y (in ten millions), for years from 2015 to 2023 are given in the following table. The variable x is the number of years after a base year of 2013.

Year	2015	2016	2017	2018	2019	2020	2021	2022	2023
x	2	3	4	5	6	7	8	9	10
y	4.23	5.01	6.62	11.92	16.71	20.21	24.01	25.08	25.18

(a) Draw a scatter diagram for these values, labelling the axes. [1]

A statistician theorises that the number of subscribers can be modelled by one of the formulae

C: $y=a\ln x+b$
D: $y=ax^2+b$

(b) Find, correct to 4 decimal places, the value of the product moment correlation coefficient

(i) between y and $\ln x$, [1]
(ii) between y and x^2. [1]

(c) Explain which model, C or D, gives a better fit to the data and find the equation of the regression line for this model. [3]

(d) Use the equation of the regression line to estimate the number of subscribers in 2024 correct to 4 significant figures and explain whether your estimate is reliable. [2]

(e) Comment on the suitability of using this model in the long run. [2]

11 **(a)** Four male students and four female students stand in two straight rows, four at the front and four at the back, to take a group photo. Among the eight of them, three of them are from the same class and all other students are from different classes.

How many ways can this be done if

(i) not all the students in the same class are standing next to one another in the same row, [3]

(ii) in each row, the boys and girls alternate? [3]

(b) A sports committee in a school comprises team leaders from four different classes. There are 6 team leaders from Class Grace, 4 team leaders from Class Hope, 5 team leaders from Class Joy and 3 team leaders from Class Piety. A teacher wants to form a working party of 7 team leaders to take charge of a carnival.

(i) Find the probability that in the party, there is at least 1 team leader from each of the 4 classes and there are more team leaders from Class Piety than from any other classes. [4]

(ii) The selected working party comprises 3 team leaders from Class Grace, 2 team leaders from Class Hope and 1 team leader each from Class Joy and Class Piety. The working party and the teacher sit around a round table in the library for discussion. Find the probability that the teacher sits in between 2 team leaders from the same class. [3]

12 In this question you should state the parameters of any distribution you use.

An electric power service company keeps records of the installation time of its electricity meters in the houses in a new estate. The time taken to install an electricity meter is normally distributed with mean 45 minutes and standard deviation 6 minutes.

(a) Sketch the distribution for the installation time of an electricity meter between 20 minutes and 70 minutes. [2]

(b) A house that has its electricity meter take more than 1 hour to be installed is considered 'inefficient'. The company randomly selects n houses in the estate for quality control in their services. Find the greatest value of n such that the probability that fewer than 3 of these n houses are 'inefficient' is at least 0.90. [3]

(c) Each month, the amount of electricity, in kilowatt-hours (kWh), used by a particular household in the estate has the distribution $\mathrm{N}(524, 27^2)$. The company charges households for electricity used at \$0.26 per kWh and each household is billed every two months. Find the probability that a randomly chosen bill for this household is more than \$270 given that it is between \$250 and \$280. State an assumption that is needed for your calculations to be valid. [4]

(d) The company also installs gas meters in the houses in the estate. The time taken in minutes to install a gas meter follows a normal distribution. 40% of the gas meters each has an installation time greater than 53 minutes and 15% of the gas meters each has an installation time less than 38 minutes. Find the mean and standard deviation of the installation times of the gas meters in the estate. [4]

Question 1 (Connected Rates of Change)

(i)	$k=r\theta+2r=r(\theta+2)\Rightarrow\theta+2=\frac{k}{r}\Rightarrow\theta=\frac{k}{r}-2$ $A=\frac{1}{2}r^2\theta=\frac{1}{2}r^2\left(\frac{k}{r}-2\right)=\frac{kr}{2}-r^2$ $\frac{dA}{dr}=\frac{k}{2}-2r$	Interestingly, many students could derive the formula $A=\frac{1}{2}r^2\theta$ by considering $A=\frac{\theta}{2\pi}\times\pi r^2$ but not many could do so similarly for the arc length formula $s=r\theta$. As a result, there are many students who could not prove the required result.
(ii)	$\frac{dA}{dt}=\left(\frac{k}{2}-2r\right)\frac{dr}{dt}$ $=\left(\frac{k}{2}-2\times\frac{k}{3}\right)\frac{k}{10}$ $=-\frac{k^2}{60}$	This part of the question is better performed than the previous part as almost the whole cohort is able to relate the rates of change. However, many could not apply the information that "arc length is equal to the radius" in a useful manner.

Question 2 (Systems of Linear Equations)

$(-2)^3+a(-2)^2+b(-2)+c=0$ $\Rightarrow -8+4a-2b+c=0$ $\Rightarrow 4a-2b+c=8$ --- (1) $\left(3e^{i\left(-\frac{2}{3}\pi\right)}\right)^3+a\left(3e^{i\left(-\frac{2}{3}\pi\right)}\right)^2+b\left(3e^{i\left(-\frac{2}{3}\pi\right)}\right)+c=0$ $27e^{i(-2\pi)}+9ae^{i\left(-\frac{4}{3}\pi\right)}+3be^{i\left(-\frac{2}{3}\pi\right)}+c=0$ $27+9a\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+3b\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+c=0$ $\left(27-\frac{9}{2}a-\frac{3}{2}b+c\right)+\left(\frac{9\sqrt{3}}{2}a-\frac{3\sqrt{3}}{2}b\right)i=0$ Comparing real parts, $27-\frac{9}{2}a-\frac{3}{2}b+c=0$ --- (2) Comparing imaginary parts, $\frac{9\sqrt{3}}{2}a-\frac{3\sqrt{3}}{2}b=0\Rightarrow 3a-b=0$ --- (3) Solving (1), (2) and (3) with the GC, we get $a=5,\ b=15,\ c=18.$ <u>Alternative Method</u> $3e^{i\left(-\frac{2}{3}\pi\right)}=3\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-\frac{3}{2}-\frac{3\sqrt{3}}{2}i$ Since all coefficients of the polynomial are real, then $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$ is also a root of the equation by the Conjugate Root Theorem. Therefore, z^3+az^2+bz+c $=(z+2)\left(z+\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)\left(z+\frac{3}{2}-\frac{3\sqrt{3}}{2}i\right)$ $=(z+2)\left(\left(z+\frac{3}{2}\right)^2+\frac{9(3)}{4}\right)$ $=(z+2)(z^2+3z+9)$ $=z^3+5z^2+15z+18$ Therefore $a=5,\ b=15,\ c=18$	Both methods were fairly common. Many careless mistakes were made in calculations. Common mistakes: × Not stating "Since the coefficients of the polynomial are all real" when using the Conjugate Root Theorem. × Incorrect conversion of complex numbers from exponential form to cartesian form. (Please identify the <u>quadrant</u> the complex number is in.) × Not converting the complex numbers to cartesian form (in the first method). × Writing "$z=3e^{i\left(-\frac{2}{3}\pi\right)}$ is a root …". (Should be "$3e^{i\left(-\frac{2}{3}\pi\right)}$ is a root …" or "$z=3e^{i\left(-\frac{2}{3}\pi\right)}$ is a solution …").

Question 3 (Inequalities)

(i) Method 1

$$\frac{x-6}{4x^2+x-5} \geq 1$$

$$\Rightarrow \frac{x-6}{4x^2+x-5} - 1 \geq 0$$

$$\Rightarrow \frac{x-6-(4x^2+x-5)}{4x^2+x-5} \geq 0$$

$$\Rightarrow \frac{-4x^2-1}{4x^2+x-5} \geq 0$$

$$\Rightarrow \frac{4x^2+1}{4x^2+x-5} \leq 0$$

$$\Rightarrow \frac{4x^2+1}{(4x+5)(x-1)} \leq 0$$

$$\Rightarrow (4x+5)(x-1) \leq 0$$

since $4x^2+1 \geq 1 > 0$ for all $x \in \mathbb{R}$

Method 2

$$\frac{x-6}{4x^2+x-5} \geq 1$$

$$\Rightarrow (x-6)(4x^2+x-5) \geq (4x^2+x-5)^2$$

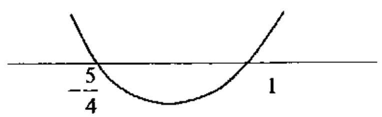
$$\Rightarrow (4x^2+x-5)\left[(x-6)-(4x^2+x-5)\right] \geq 0$$

$$\Rightarrow (4x^2+x-5)(-4x^2-1) \geq 0$$

$$\Rightarrow (4x+5)(x-1)(-4x^2-1) \geq 0$$

$$\Rightarrow (4x+5)(x-1) \leq 0$$

since $-4x^2-1 \leq 1 < 0$ for all $x \in \mathbb{R}$



Therefore, $-\frac{5}{4} < x < 1$ since $x \neq -\frac{5}{4}$ and $x \neq 1$

Many candidates correctly simplified the inequality to $\frac{-4x^2-1}{4x^2+x-5} \geq 0$.

The correct factorisation of $4x^2+x-5$ into $(4x+5)(x-1)$ is often seen as well.

Some candidates made mistakes in the algebraic manipulations e.g. writing the numerator as $4x^2-1$.

There are some candidates who tried to factorise $4x^2+1$. Please note that complex roots are not considered for the determination of the critical values.

Common mistakes:

- × Inclusion of $x = -\frac{5}{4}$ and $x = 1$ in the solution of the inequality. These values make the denominator in the original inequality 0, so they need to be excluded.
- × Incorrect evaluation of signs in the number line, leading to incorrect range of solutions $x < -\frac{5}{4}$ or $1 < x$

(ii) $\frac{x-6x^2}{4+x-5x^2} \geq 1$

Replace x with $\frac{1}{y}$ and we get

$$\frac{\frac{1}{y}-6\left(\frac{1}{y}\right)^2}{4+\frac{1}{y}-5\left(\frac{1}{y}\right)^2} \geq 1 \Rightarrow \frac{\left(\frac{y-6}{y^2}\right)}{\left(\frac{4y^2+y-5}{y^2}\right)} \geq 1 \Rightarrow \frac{y-6}{4y^2+y-5} \geq 1$$

Therefore, $-\frac{5}{4} < y < 1$, i.e. $-\frac{5}{4} < \frac{1}{x} < 1$.

So

$-\frac{5}{4} < \frac{1}{x} < 0$ or $0 < \frac{1}{x} < 1$

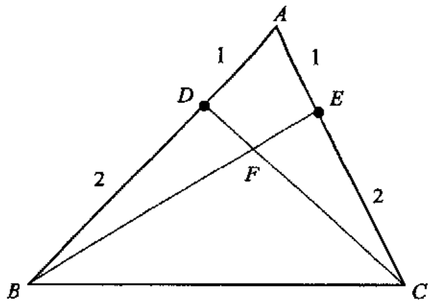
$x < -\frac{4}{5}$ or $x > 1$

Many students are able to identify that a suitable replacement is $\frac{1}{y}$.

Common mistakes:

- × Solving $-\frac{5}{4} < \frac{1}{x} < 1$ is not simply just taking the reciprocals of the terms and keeping the sign i.e., the inequality above is **not equivalent** to $-\frac{4}{5} < x < \frac{1}{1}$. A reliable way to solve reciprocals is to look at the graph $y = \frac{1}{x}$.
- × Some students wrote '$x = \frac{1}{x}$' to mean 'replace x with $\frac{1}{x}$'. If the former holds, then $x = \pm 1$, which is not part of the solution set.

Question 4 (Vectors I)



Using Ratio Theorem,

$\overrightarrow{BE}$

$=\frac{1}{3}\left(\overrightarrow{BC}+2\overrightarrow{BA}\right)$

$=\frac{1}{3}\left(\mathbf{c}-\mathbf{b}+2(\mathbf{a}-\mathbf{b})\right)$

$=\frac{1}{3}\mathbf{c}+\frac{2}{3}\mathbf{a}-\mathbf{b}$

$\overrightarrow{CD}$

$=\frac{1}{3}\left(\overrightarrow{CB}+2\overrightarrow{CA}\right)$

$=\frac{1}{3}\left(\mathbf{b}-\mathbf{c}+2(\mathbf{a}-\mathbf{c})\right)$

$=\frac{1}{3}\mathbf{b}+\frac{2}{3}\mathbf{a}-\mathbf{c}$

$l_{BE}:\mathbf{r}=\mathbf{b}+\lambda\left(\frac{1}{3}\mathbf{c}+\frac{2}{3}\mathbf{a}-\mathbf{b}\right)$

$\quad=\frac{2}{3}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}+\frac{1}{3}\lambda\mathbf{c},\ \lambda\in\mathbb{R}$ (shown)

$l_{CD}:\mathbf{r}=\mathbf{c}+\mu\left(\frac{1}{3}\mathbf{b}+\frac{2}{3}\mathbf{a}-\mathbf{c}\right)$

$=\frac{2}{3}\mu\mathbf{a}+\frac{1}{3}\mu\mathbf{b}+(1-\lambda)\mathbf{c},\ \mu\in\mathbb{R}$ (shown)

At F, $\frac{2}{3}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}+\frac{1}{3}\lambda\mathbf{c}=\frac{2}{3}\mu\mathbf{a}+\frac{1}{3}\mu\mathbf{b}+(1-\lambda)\mathbf{c}$

$\left(\frac{2}{3}\lambda-\frac{2}{3}\mu\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu\right)\mathbf{b}+\left(\frac{1}{3}\lambda+\mu-1\right)\mathbf{c}=\mathbf{0}$

Many students could find the vectors $\overline{BE}$ and $\overline{CD}$. However, some used vector addition to find the 2 vectors, which is very tedious.

Common mistakes:

× Some students do not know the difference between the direction vector $\overline{BE}$ and equation of the line BE.

× Some students regarded **a**, **b**, **c** as vectors **i**, **j**, **k** and wrote $\frac{1}{3}\mathbf{c}+\frac{2}{3}\mathbf{a}-\mathbf{b}=\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ -1\end{pmatrix}$

× Many students did not understand the requirements of the question. They compared the coefficients of non-parallel vectors **a**, **b**, **c** at this stage: $\frac{2}{3}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}+\frac{1}{3}\lambda\mathbf{c}$ $=\frac{2}{3}\mu\mathbf{a}+\frac{1}{3}\mu\mathbf{b}+(1-\lambda)\mathbf{c}$

As a result, they failed to obtain the required expression.

(ii) If $OACB$ is a parallelogram,

$\mathbf{c}=\mathbf{a}+\mathbf{b}$

$\left(\frac{2}{3}\lambda-\frac{2}{3}\mu\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu\right)\mathbf{b}+\left(\frac{1}{3}\lambda+\mu-1\right)\mathbf{c}=\mathbf{0}$

$\left(\frac{2}{3}\lambda-\frac{2}{3}\mu\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu\right)\mathbf{b}+\left(\frac{1}{3}\lambda+\mu-1\right)(\mathbf{a}+\mathbf{b})=\mathbf{0}$

$\left(\frac{2}{3}\lambda-\frac{2}{3}\mu+\frac{1}{3}\lambda+\mu-1\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu+\frac{1}{3}\lambda+\mu-1\right)\mathbf{b}=\mathbf{0}$

$\left(\lambda+\frac{1}{3}\mu-1\right)\mathbf{a}+\left(\frac{2}{3}\mu-\frac{2}{3}\lambda\right)\mathbf{b}=\mathbf{0}$

Since **a** and **b** are not parallel to each other and non-zero vectors,

$\lambda+\frac{1}{3}\mu-1=0 \quad\cdots(1)$

$\frac{2}{3}\mu-\frac{2}{3}\lambda=0 \quad\cdots(2)$

Solving, $\lambda=\mu=\frac{3}{4}$

Thus position vector of F is given by

$=\mathbf{b}+\frac{3}{4}\left(\frac{1}{3}(\mathbf{a}+\mathbf{b})+\frac{2}{3}\mathbf{a}-\mathbf{b}\right)$

$=\frac{3}{4}\mathbf{a}+\frac{1}{2}\mathbf{b}$

This part was usually either not attempted or very badly done. Most students could not recognise that $\mathbf{c}=\mathbf{a}+\mathbf{b}$.

Common mistakes:

× Many did not read the question and stopped at the step $\left(\lambda+\frac{1}{3}\mu-1\right)\mathbf{a}+\left(\frac{2}{3}\mu-\frac{2}{3}\lambda\right)\mathbf{b}=\mathbf{0}$

× Many used $\mathbf{c}=\mathbf{a}-\mathbf{b}$ instead.

Question 5 (Maxima & Minima Problems)

<table>
<tr><td>Let V_h and v_h be the volumes of the original cone and cone that was removed. Since both cones are similar, $\frac{v_h}{V_h} = \left(\frac{ar}{r}\right)^3 = a^3$.

Also, it is given that $h = \sqrt{4^2 - r^2}$. Hence

$V = \pi(ar)^2 r + (V_h - v_h)$
$= \pi a^2 r^3 + (V_h - a^3 V_h)$
$= \pi a^2 r^3 + (1-a^3)\left(\frac{1}{3}\pi r^2 h\right)$
$= \pi a^2 r^3 + \frac{(1-a^3)}{3}\pi r^2\sqrt{16-r^2}$</td><td>This part was very well done. However, most students did not leave their answers in simplied form.</td></tr>
</table>

<table>
<tr><td>(a)</td><td>Given that $a = 0.25$,
$V = 0.0625\pi r^3 + 0.328125\pi r^2\sqrt{16-r^2}$

$\frac{dV}{dr} = 0.1875\pi r^2 + 0.328125\pi\left[2r\sqrt{16-r^2} + \frac{r^2(-2r)}{2\sqrt{16-r^2}}\right]$
$= 0.1875\pi r^2 + 0.328125\pi\left(\frac{2r(16-r^2)-r^3}{\sqrt{16-r^2}}\right)$
$= 0.1875\pi r^2 + 0.328125\pi r\left(\frac{32-3r^2}{\sqrt{16-r^2}}\right)$

For stationary V, $\frac{dV}{dr} = 0$
$0.1875\pi r^2 + 0.328125\pi r\left(\frac{32-3r^2}{\sqrt{16-r^2}}\right) = 0$
Since $0 < r < 4$,
$r + \frac{7}{4}\left(\frac{32-3r^2}{\sqrt{16-r^2}}\right) = 0$
$\frac{7}{4}\left(\frac{3r^2-32}{\sqrt{16-r^2}}\right) = r$
$\frac{49}{16}\left(\frac{9r^4-192r^2+1024}{16-r^2}\right) = r^2$
$49(9r^4-192r^2+1024) = 256r^2 - 16r^4$
$457r^4 - 9664r^2 + 50176 = 0$</td><td>Most students could apply the Product rule but experienced great difficulty in simplifying the expression to show the required equation. Only a very small handful of students managed to show the expression correctly.</td></tr>
<tr><td>(b)</td><td>Using G.C, since $r > 0$,
$r = 3.462322463$ or $r = 3.02637266$
When $r = 3.462322463$, $\frac{dV}{dr} = 0$.
When $r = 3.02637266$, $\frac{dV}{dr} = 10.790112 \neq 0$

$h = \sqrt{16 - 3.462322463^2} = 2.00$ (to 3 s.f.)</td><td>Many did not attempt this part or tried to and obtained the correct value of r but did not check the corresponding value of the derivative.</td></tr>
</table>

Question 6 (Complex Numbers – Geometric Forms)

(i) $\frac{e^{i\theta}}{1-e^{i\theta}}=\frac{e^{i\theta}}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)}$ $=\frac{e^{i\frac{\theta}{2}}}{\left(-2\sin\frac{\theta}{2}\right)i}$ $=\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\left(-2\sin\frac{\theta}{2}\right)i}$ $=-\frac{1}{2i}\cot\frac{\theta}{2}-\frac{1}{2}$ $=\left(\frac{1}{2}\cot\frac{\theta}{2}\right)i-\frac{1}{2}$ $=\frac{1}{2}\left(i\cot\frac{\theta}{2}-1\right)$	Many students could not answer this part correctly. Most attempted to introduce $\frac{\theta}{2}$ by using half angle formula on $\cos\theta$ and $\sin\theta$ and could not proceed with meaningful simplification of the terms. **Learning points:** - Remember to try to factorise $e^{i\frac{\theta}{2}}$ first before other approaches when dealing with half angles in complex numbers setting.
(ii) $i=e^{i\frac{\pi}{2}}$ or $e^{i\left(-\frac{3\pi}{2}\right)}$ or $e^{i\left(\frac{5\pi}{2}\right)}$	Most students could give at least two different representations. $e^{i\left(-\frac{3\pi}{2}\right)}$ is the representation that are missed the most by the candidates.

(iii) $\left(\frac{w}{w+1}\right)^3=i=e^{i\frac{\pi}{2}}$ or $e^{i\left(-\frac{3\pi}{2}\right)}$ or $e^{i\left(\frac{5\pi}{2}\right)}$ by **(ii)** $\Rightarrow\frac{w}{w+1}=e^{i\frac{\pi}{6}}$ or $e^{i\left(-\frac{\pi}{2}\right)}$ or $e^{i\left(\frac{5\pi}{6}\right)}$ Now let $\frac{w}{w+1}=e^{i\theta}$, where $\theta=-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$. Then we have $w=e^{i\theta}w+e^{i\theta}$ $\Rightarrow w\left(1-e^{i\theta}\right)=e^{i\theta}$ $\Rightarrow w=\frac{e^{i\theta}}{1-e^{i\theta}}=\frac{1}{2}\left(i\cot\frac{\theta}{2}-1\right)$ by **(i)** Therefore $w=\frac{1}{2}\left(i\cot\left(-\frac{\pi}{4}\right)-1\right),\ \frac{1}{2}\left(i\cot\left(\frac{\pi}{12}\right)-1\right)$ or $\frac{1}{2}\left(i\cot\left(\frac{5\pi}{12}\right)-1\right)$.	Most candidates did not write any useful working for this part. **Learning points:** - Exponential forms of complex number follows the exponential rules e.g., $z^3=e^{i\alpha}$ implies $z=e^{i\frac{\alpha}{3}}$. Only a small number of candidates could make the connection with the earlier part and manipulated to obtain $w=\frac{e^{i\theta}}{1-e^{i\theta}}=\frac{1}{2}\left(i\cot\frac{\theta}{2}-1\right)$ Of which some did not divide the angle obtained $\theta=-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ by 2 in the final solution.

Question 7 (Power Series)

(i)

Method 1

$y=\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{d^2y}{dx^2}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\left(-2\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)\right)$

$\quad -2\cot\left(2x+\frac{\pi}{4}\right)\left(-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)\right)$

$\quad =4y^3+4y\cot^2\left(2x+\frac{\pi}{4}\right)$

$\quad =4y^3+4y\left(y^2-1\right)$

$\quad =8y^3-4y$ (shown)

Method 2

$y=\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx}=-2y\cot\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{d^2y}{dx^2}=-2\frac{dy}{dx}\cot\left(2x+\frac{\pi}{4}\right)-2y\left(-2\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)\right)$

$\Rightarrow \frac{d^2y}{dx^2}=4y\cot^2\left(2x+\frac{\pi}{4}\right)+4y^3$

$\Rightarrow \frac{d^2y}{dx^2}=4y\left(\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)-1\right)+4y^3$

$\Rightarrow \frac{d^2y}{dx^2}=4y\left(y^2-1\right)+4y^3$

$\Rightarrow \frac{d^2y}{dx^2}=4y^3-4y+4y^3=8y^3-4y$ (shown)

Common errors:

- Many students did not apply the relevant formula in MF26 to differentiate $\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$.

  They resorted to converting the expression into '1/sin…' and used the compound angle formula to unnecessarily expand the function and this resulted in tedious work for finding the second derivative in the later part and thus little success to prove the result.

- Similarly, some students did not know how to differentiate $\cot\left(2x+\frac{\pi}{4}\right)$. They changed it to $\frac{1}{\tan\left(2x+\frac{\pi}{4}\right)}$ and differentiated the function using the Chain rule. A few **incorrectly** thought that $\frac{1}{\tan x}=\tan^{-1}x$.

- did not apply Chain Rule fully, miss out the constant and negative sign

- did not apply $1+\cot^2 x=\operatorname{cosec}^2 x$

Learning points: Please

- familiarise yourselves with what formula are given in MF26.
- memorise formulas for differentiating and integrating trigonometric functions and trigonometric identities that are not in MF26.
- apply implicit differentiation wherever possible for this type of questions in Power Series

Method 3

$y=\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx}=-2y\sqrt{y^2-1}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2=4y^2\left(y^2-1\right)=4y^4-4y^2$

$\Rightarrow 2\frac{dy}{dx}\frac{d^2y}{dx^2}=\left(16y^3-8y\right)\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=8y^3-4y$ (shown)

Method 4

$\frac{1}{y}=\sin\left(2x+\frac{\pi}{4}\right)$ (1)

$-\frac{1}{y^2}\frac{dy}{dx}=2\cos\left(2x+\frac{\pi}{4}\right)$

$\frac{dy}{dx}=-2y^2\cos\left(2x+\frac{\pi}{4}\right)$ (2)

$\frac{d^2y}{dx^2}=-4y\frac{dy}{dx}\cos\left(2x+\frac{\pi}{4}\right)+4y^2\sin\left(2x+\frac{\pi}{4}\right)$

$=8y^3\cos^2\left(2x+\frac{\pi}{4}\right)+4y^2\sin\left(2x+\frac{\pi}{4}\right)$ (from (2))

$=8y^3\left(1-\sin^2\left(2x+\frac{\pi}{4}\right)\right)+4y^2\sin\left(2x+\frac{\pi}{4}\right)$

$=8y^3\left(1-\frac{1}{y^2}\right)+4y^2\cdot\frac{1}{y}$ (from (1))

$=8y^3-8y+4y$

$=8y^3-4y$ (shown)

(ii) $\dfrac{d^2y}{dx^2}=8y^3-4y \Rightarrow \dfrac{d^3y}{dx^3}=24y^2\dfrac{dy}{dx}-4\dfrac{dy}{dx}$

When $x=0$,

$y=\operatorname{cosec}\left(\frac{\pi}{4}\right)=\sqrt{2}$

$\dfrac{dy}{dx}=-2\operatorname{cosec}\left(\frac{\pi}{4}\right)\cot\left(\frac{\pi}{4}\right)=-2\sqrt{2}$

$\dfrac{d^2y}{dx^2}=8\left(\sqrt{2}\right)^3-4\sqrt{2}=12\sqrt{2}$

$\dfrac{d^3y}{dx^3}=24(2)\left(-2\sqrt{2}\right)-4\left(-2\sqrt{2}\right)=-88\sqrt{2}$

Hence

$$\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\approx\sqrt{2}+\left(-2\sqrt{2}\right)\frac{x}{1!}+\left(12\sqrt{2}\right)\frac{x^2}{2!}+\left(-88\sqrt{2}\right)\frac{x^3}{3!}$$
$$=\sqrt{2}-2\sqrt{2}x+6\sqrt{2}x^2-\frac{44}{3}\sqrt{2}x^3$$

Common errors:
- obtained $\dfrac{d^3y}{dx^3}=24y^2-4$ or differentiated $8y^3$ w.r.t x wrongly.
- did not simplify coefficient to $k\sqrt{2}$
- a few wrote the Maclaurin series as a sum of powers of $\left(x+\frac{\pi}{4}\right)$

Learning points:
- Maclaurin's Theorem can be referenced from MF26 formula.
- Show working to find the values of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$ when $x=0$ **first, before** substituting the values into the Maclaurin series coefficients. Doing these simultaneously usually results in arithmetic errors.

(iii) Differentiating the expansion above, we get

$$-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)\approx-2\sqrt{2}+12\sqrt{2}x-44\sqrt{2}x^2.$$

Therefore

$$\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)\approx\sqrt{2}-6\sqrt{2}x+22\sqrt{2}x^2.$$

Let $2x+\dfrac{\pi}{4}=\dfrac{13\pi}{50}$. Then $x=\dfrac{\pi}{200}$.

Hence

$$\operatorname{cosec}\left(\frac{13\pi}{50}\right)\cot\left(\frac{13\pi}{50}\right)\approx\sqrt{2}\left(1-6\left(\frac{\pi}{200}\right)+22\left(\frac{\pi}{200}\right)^2\right)$$
$$\approx\sqrt{2}\left(1-\frac{3}{100}\pi+\frac{11}{20000}\pi^2\right)$$

Poorly attempted. Many students left this part blank or just substituted $x=\pi/200$ or $x=\pi/100$ into their answer in part **(ii)** and consequently were stuck with $\cot(x+\pi/4)$.

Learning points:
- 'Hence' implies there is a need to apply the results from the previous parts. Observe that $\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)\approx\left(-\frac{1}{2}\right)\dfrac{dy}{dx}$.
- Differentiate the answer in part **(ii)** w.r.t x first before substituting $x=\pi/200$. Students who attempted to do these together in one step usually made arithmetic errors.

Question 8 (Sequences and Series)

(a) For $n\ge 2$,

$$u_n=An^2+Bn+2^{n+1}-\left(A(n-1)^2+B(n-1)+2^n\right)$$
$$=An^2+Bn+2^{n+1}-\left(An^2-2An+A+Bn-B+2^n\right)$$
$$=2(2^n)+2An-A+B-2^n=2^n+(2n-1)A+B$$

$u_3=2^3+(2(3)-1)A+B=21\Rightarrow 5A+B=13\cdots(1)$

$u_5=2^5+(2(5)-1)A+B=53\Rightarrow 9A+B=21\cdots(2)$

Solving (1) and (2). $A=2$ and $B=3$. Hence

$$u_n=\begin{cases}2(1)^2+3(1)+2^{1+1}, & n=1\\ 2^n+4n+1, & n\ge 2\end{cases}=\begin{cases}9, & n=1\\ 2^n+4n+1, & n\ge 2\end{cases}$$

Common errors:
- take $S_3=21$ and $S_5=53$
- did not simplify $2^{n+1}-2^n=2\left(2^n\right)-2^n=2^n$
- tried to apply AP/GP formula
- wrote $u_n=S_{n+1}-S_n$
- did not check for u_1 and S_1. Only a handful of students got the correct answer for u_n.

Learning points:

To find u_n from S_n, use $u_n=S_n-S_{n-1}$, $n\ge 2$ (*). Please check if the formula (*) is consistent with (i.e. is equal to) S_n for $n=1$. If it is not, you need to write u_n as a piecewise function of n, and write $u_1=S_1$ separately. If it is, the answer can be written as a single function $u_n=S_n-S_{n-1}$, $n\ge 1$.

(b)(i)

$$\sum_{r=1}^{n}\ln\left(\frac{r(r+2)}{(r+1)^2}\right)=\sum_{r=1}^{n}\left(\ln\left(\frac{r+2}{r+1}\right)-\ln\left(\frac{r+1}{r}\right)\right)$$
$$=\left(\ln\left(\frac{3}{2}\right)-\ln 2\right)$$
$$+\left(\ln\left(\frac{4}{3}\right)-\ln\left(\frac{3}{2}\right)\right)$$
$$\vdots$$
$$+\left(\ln\left(\frac{n+1}{n}\right)-\ln\left(\frac{n}{n-1}\right)\right)$$
$$+\left(\ln\left(\frac{n+2}{n+1}\right)-\ln\left(\frac{n+1}{n}\right)\right)$$
$$=-\ln 2+\ln\left(\frac{n+2}{n+1}\right)$$
$$=\ln\left(\frac{n+2}{n+1}\right)-\ln 2 \text{ (shown)}$$

Common errors:
- could not apply laws of logarithm to simplify expression into the form $\ln r-2\ln(r+1)+\ln(r+2)$ or $\ln\left(\frac{r+2}{r+1}\right)-\ln\left(\frac{r+1}{r}\right)$ or $\ln\left(\frac{r}{r+1}\right)-\ln\left(\frac{r+1}{r+2}\right)$ that would allow for the method of differences to be applied.
- rewrote as $\ln(r^2+2r)-\ln(r+1)^2$ or attempted to use partial fractions, with little success.
- did not put brackets properly, please note that $\sum_{r=1}^{n}\left(\ln\left(\frac{r+2}{r+1}\right)-\ln\left(\frac{r+1}{r}\right)\right)\ne\sum_{r=1}^{n}\ln\left(\frac{r+2}{r+1}\right)-\ln\left(\frac{r+1}{r}\right)$
- not cancelling the terms properly (or not cancelling any single term). Not showing at least one full cancellation above and at least one full cancellation below.

Alternative Solution

$$\sum_{r=1}^{n}\ln\left(\frac{r(r+2)}{(r+1)^2}\right)=\sum_{r=1}^{n}\left(\ln r-2\ln(r+1)+\ln(r+2)\right)$$

$=(\ln 1-2\ln(2)+\ln(3))$
$+(\ln 2-2\ln(3)+\ln(4))$
$+(\ln 3-2\ln(4)+\ln(5))$
$\vdots$
$+(\ln(n-2)-2\ln(n-1)+\ln(n))$
$+(\ln(n-1)-2\ln(n)+\ln(n+1))$
$+(\ln(n)-2\ln(n+1)+\ln(n+2))$
$=\ln 1-\ln 2-\ln(n+1)+\ln(n+2)$
$=\ln(n+2)-\ln(n+1)-\ln 2$
$=\ln\left(\frac{n+2}{n+1}\right)-\ln 2$ (shown)

Learning points:
- For the alternative solution, you need to write 1st 3 rows and last 3 rows so that one full cancellation above and one full cancellation below are shown.
- For "Show" type of MOD question, you need to write out the terms after cancellation before the final shown answer.

(b)(ii)

$$\sum_{r=0}^{n}\ln\left(\frac{r^2+4r+3}{(r+2)^2}\right)=\sum_{r=0}^{n}\ln\left(\frac{(r+1)(r+3)}{(r+2)^2}\right)$$
$$=\sum_{q-1=0}^{q-1=n}\ln\left(\frac{q(q+2)}{(q+1)^2}\right)$$
$$=\sum_{q=1}^{q=n+1}\ln\left(\frac{q(q+2)}{(q+1)^2}\right)$$
$$=\ln\left(\frac{n+1+2}{n+1+1}\right)-\ln 2$$
$$=\ln\left(\frac{n+3}{n+2}\right)-\ln 2$$

Common errors:
- Many could not perform the appropriate substitution to **manipulate the current series to "appear" like the previous series.**
- Many started with the series in part **(b)(i)** and could not proceed or could not get the correct answer.

Learning points:
- As the general term is different from the series in part **(b)(i)**, the correct technique is to use the substitution method.
- Starting from the current series, replace r with $q-1$, change the general term to that in part **(b)(i)** and then replace the lower and upper limit with $q-1=0$ and $q-1=n$.

Question 9 (Curve Sketching)

(a) $y=\left(x^2+cx\right)e^{-x}$

$\frac{dy}{dx}=(2x+c)e^{-x}-\left(x^2+cx\right)e^{-x}$
$=\left(-x^2-cx+2x+c\right)e^{-x}$
$=-\left(x^2+(c-2)x-c\right)e^{-x}$

At stationary points, $\frac{dy}{dx}=0$.

$-\left(x^2+(c-2)x-c\right)e^{-x}=0$
$x^2+(c-2)x-c=0$ ($\because e^{-x}>0$ for all $x\in\mathbb{R}$).

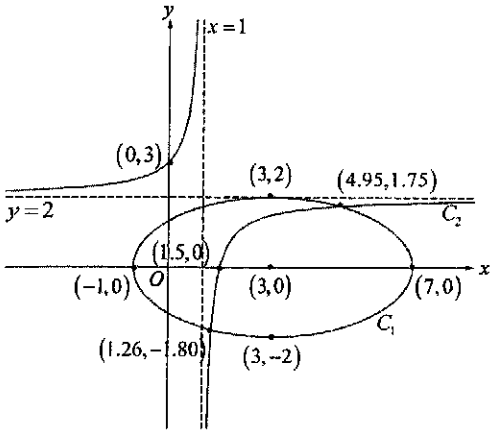
Discriminant, $D=(c-2)^2-4(1)(-c)$
$=c^2-4c+4+4c$
$=c^2+4\geq 4>0$ for all $c\in\mathbb{R}$.

$\therefore$ The equation $\frac{dy}{dx}=0$ has two real and distinct roots.

Therefore, the curve with equation $y=\left(x^2+cx\right)e^{-x}$ has two stationary points for all real values of c.

Please note that in questions that require a proof, it is important to demonstrate your reasoning and justify steps where necessary.

Common mistakes:
- ×not stating "$e^{-x}>0$ (or $e^{-x}\neq 0$) for all $x\in\mathbb{R}$" in the proof at the step of dividing throughout by e^{-x}.
- ×Solving for the roots and stopping short of explaining why both roots are **real**.
- ×computing the discriminant of $\left(x^2+(c-2)x-c\right)e^{-x}$. This is incorrect because the expression above is **NOT** a quadratic expression in x, so it is meaningless to talk about the discriminant of $\left(x^2+(c-2)x-c\right)e^{-x}$.
- ×writing the discriminant as "b^2-4ac" when c is already used in the question.
- ×writing $c^2>0$ for all real values of c. The correct inequality is $c^2\geq 0$ for all real values of c.
- ×assuming that "discriminant >0" straightaway. This is incorrect because you are assuming precisely what you need to prove.
- ×trying to show that "discriminant ≥ 0" instead of "discriminant >0". Recall your O-Level content on discriminants.
- ×It is appalling that a notable number of students expanded $(c-2)^2$ as c^2-2c+4.

(b) For C_1: $x^2+4y^2-6x-7=0$ $x^2-6x+9-9+4y^2=7$ $(x-3)^2+4y^2=16$ $\frac{(x-3)^2}{16}+\frac{y^2}{4}=1$ For C_2: $y=\frac{2x-3}{x-1}$ $=2-\frac{1}{x-1}$ 	C_2 was generally well-sketched, but many mistakes were made in the sketch of C_1. Common mistakes: × completing the square incorrectly. Please note that there is no y term in the equation of the ellipse. × missing labels such as coordinates of centre and vertices of the ellipse, points of intersection between the curves, equations of the curves (or at least C_1 and C_2), and the origin. × The positions of C_1 and C_2 relative to each other are not accurate. For example, the top of the ellipse not touching the asymptote $y=2$ at one point, or the point $(1.5,0)$ is too near or to the right of the point $(3,0)$. × sketching a hyperbola instead of an ellipse.

Question 10 (Functions)

(i) For $-4<x<-1$, $f(x)=-(x+1)^2$ Let $y=-(x+1)^2$. $(x+1)^2=-y$ $x+1=\pm\sqrt{-y}$ $x=-1\pm\sqrt{-y}$ Since $-4<x<-1$, $x=-1-\sqrt{-y}$ For $-1\le x\le 2$, $f(x)=(x+1)^2$ Let $y=(x+1)^2$. $(x+1)^2=y$ $x+1=\pm\sqrt{y}$ $x=-1\pm\sqrt{y}$ Since $-1\le x\le 2$, $x=-1+\sqrt{y}$ Thus, $f^{-1}(x)=\begin{cases}-1-\sqrt{-x}, & \text{for } x\in\mathbb{R}, -9<x<0,\\ -1+\sqrt{x}, & \text{for } x\in\mathbb{R}, 0\le x\le 9.\end{cases}$	Many assumed that $\sqrt{-y}$ should be rejected, not realising that y **can be negative**. Hence many omitted the rule $-1-\sqrt{-x}$. Unfortunately, there are still a lot of students who did not consider $\pm\sqrt{\ }$ and did not use the domain of the function to determine the correct square root. **The skill of choosing the correct square root has been tested so many times in many topics** and thus it is expected that candidates should have a good grasp of this skill by now. Another group of students recalled the relevance of $\pm$ but were unsure where it should be placed and hence modulus signs were used everywhere without proper consideration. Examples include $-1\pm\sqrt{\lvert x\rvert}$, $\sqrt{\lvert x\rvert}\pm 1$, $1+\lvert\sqrt{x}\rvert$ etc. Domains of the inverse were NOT properly thought through, with many thinking that the value -1 was relevant since the rule was $-1\pm\sqrt{\ldots}$

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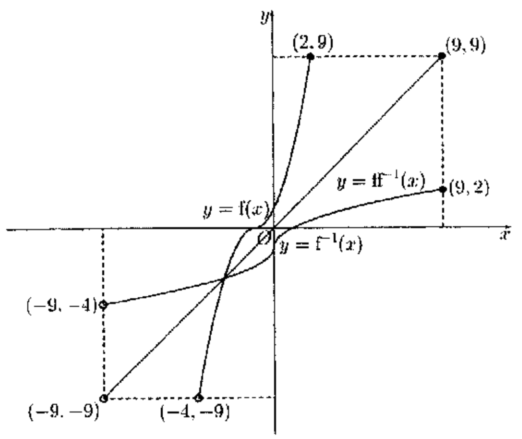
(ii)		For such questions, details are necessary, including **domains, nature of turning points, inclusion and exclusion of points and symmetrical properties.** Many did not sketch the graphs to scale and hence made the diagram awkward-looking. The composite function $y=\mathrm{ff}^{-1}(x)$ was poorly sketched with many not paying attention to the domain. This type of questions has appeared many times in practice questions and revision. Thus, it is expected that candidates should not be scoring 2 marks or less for such questions.
(iii)	For $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, $\mathrm{f}(x)=x$. From the graph, intersection occurs for $-4<x\le -1$. Thus $-(x+1)^2=x$ $x^2+2x+1=-x$ $x^2+3x+1=0$ $x=\dfrac{-3\pm\sqrt{3^2-4}}{2}$ $x=\dfrac{-3\pm\sqrt{5}}{2}$ $x=\dfrac{-3-\sqrt{5}}{2}$ $(\because x<-1)$ Thus, for $\mathrm{f}(x)\le \mathrm{f}^{-1}(x)$, $-4<x\le\dfrac{-3-\sqrt{5}}{2}$.	The use of calculator is not allowed since there is an 'exact' requirement. Unfortunately, candidates still left their answers in non-exact form. The skill of equating $\mathrm{f}(x)$ to x to solve the inequality is also available in practice questions. Candidates should use the graph to decide the correct rule of $\mathrm{f}(x)$ to equate to x. If so, at least 2 marks would have been easily obtained.

Question 11 (Differential Equations)

(i)	Since $p=\dfrac{1}{k}\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=kp$. $\therefore \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=k\dfrac{\mathrm{d}p}{\mathrm{d}x}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=ak\sqrt{1+\left(\dfrac{1}{k}\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ $\Rightarrow k\dfrac{\mathrm{d}p}{\mathrm{d}x}=ak\sqrt{1+p^2}$ $\Rightarrow \dfrac{\mathrm{d}p}{\mathrm{d}x}=a\sqrt{1+p^2}$ (shown)	Some students were confused about what to do here. A number of students skipped critical steps.
(ii)	$p=\tan u\Rightarrow \dfrac{\mathrm{d}p}{\mathrm{d}x}=\dfrac{\mathrm{d}p}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}x}=\left(\sec^2 u\right)\dfrac{\mathrm{d}u}{\mathrm{d}x}$. Thus, $\dfrac{\mathrm{d}p}{\mathrm{d}x}=a\sqrt{1+p^2}$ $\Rightarrow\left(\sec^2 u\right)\dfrac{\mathrm{d}u}{\mathrm{d}x}=a\sqrt{1+\tan^2 u}$ $\Rightarrow\left(\sec^2 u\right)\dfrac{\mathrm{d}u}{\mathrm{d}x}=a\sec u$ [**Remember that $\sec^2 u=1+\tan^2 u$.**] [**Since the DE is of the form $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{g}(u)$, simplify as $\dfrac{1}{\mathrm{g}(u)}\dfrac{\mathrm{d}u}{\mathrm{d}x}=1\Rightarrow\displaystyle\int\frac{1}{\mathrm{g}(u)}\,\mathrm{d}u=\int 1\,\mathrm{d}x$.**] $\Rightarrow\sec u\dfrac{\mathrm{d}u}{\mathrm{d}x}=a$ $\Rightarrow\displaystyle\int\sec u\,\mathrm{d}u=\int a\,\mathrm{d}x$ [**Integral formula for the secant function is in MF26.**] $\Rightarrow\ln\left(\sec u+\tan u\right)=ax+c$ $\because \lvert u\rvert<\dfrac{\pi}{2}$ $\Rightarrow\ln\left(\sqrt{1+p^2}+p\right)=ax+c$ (Note: $p=\tan u\Rightarrow\sec^2 u=1+p^2$) $\Rightarrow\sqrt{1+p^2}+p=\mathrm{e}^{ax+c}$ $\Rightarrow\sqrt{1+p^2}+p=A\mathrm{e}^{ax}$, $A=\mathrm{e}^{c}$	Many students were not familiar with the procedure to solve a differential equation via a given substitution. For some, their reductions even led to an equation that was void of any derivative. Please revise. Some students could not apply the technique needed to solve this DE (method of separation). Workings such as $p=\displaystyle\int\sqrt{1+p^2}\,\mathrm{d}x$ and $u=\displaystyle\int\sec u\,\mathrm{d}x$ were fairly common. Some students approached the question as one on integration by substitution, which is acceptable. However, many of these students simplified $\displaystyle\int\frac{1}{\sqrt{1+p^2}}\,\mathrm{d}p$ incorrectly to $\displaystyle\int\frac{1}{\sqrt{1+\tan^2 u}}\times\boxed{\frac{1}{\sec^2 u}}\,\mathrm{d}u$. From MF26, if $\lvert u\rvert<\dfrac{\pi}{2}$, $\int\sec u\,\mathrm{d}u$ is equals to $\ln(\sec u+\tan u)+c$ aka no modulus is required. Working with modulus should be as follows: $\displaystyle\int\sec u\,\mathrm{d}u=\int a\,\mathrm{d}x$ $\Rightarrow\ln\lvert\sec u+\tan u\rvert=ax+c$ $\Rightarrow\lvert\sec u+\tan u\rvert=\mathrm{e}^{ax+c}$ $\Rightarrow\sec u+\tan u=\pm\mathrm{e}^{ax+c}=\pm\mathrm{e}^{c}\mathrm{e}^{ax}$ $\Rightarrow\sec u+\tan u=A\mathrm{e}^{ax}$, $A=\pm\mathrm{e}^{c}$ Some students had no modulus but still had a ± later. This is incorrect.

(ii) Since the turning point lies on the y-axis, when $x=0$, $p=\frac{1}{k}\frac{dy}{dx}=0$. Hence, $\sqrt{1+0^2}+0=Ae^0 \Rightarrow A=1$. So,

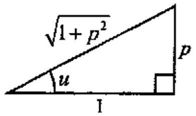
$$\sqrt{1+p^2}+p=e^{ax} \Rightarrow \sqrt{1+p^2}=e^{ax}-p$$

$$\Rightarrow 1+p^2=\left(e^{ax}-p\right)^2=e^{2ax}-2pe^{ax}+p^2$$

$$\Rightarrow 2pe^{ax}=e^{2ax}-1$$

$$\Rightarrow p=\frac{e^{2ax}-1}{2e^{ax}}=\frac{e^{ax}-e^{-ax}}{2} \text{ (shown)}$$

Some students could not convert $\sec u$ to $\sqrt{1+p^2}$. Please use trigonometric identities or the right-angled triangle method:



$$\sec u=\frac{\text{hyp}}{\text{adj}}=\sqrt{1+p^2}$$

To simplify an equation comprising a surd, isolate the surd term on its own on one side of the equation first, **before** squaring both sides.

(iii)

$$\frac{1}{0.701}\frac{dy}{dx}=\frac{e^{-0.0329x}-e^{0.0329x}}{2}$$

$$y=0.701\int\frac{e^{-0.0329x}-e^{0.0329x}}{2}\,dx$$

$$=\frac{0.701}{2}\left(\frac{e^{-0.0329x}}{-0.0329}-\frac{e^{0.0329x}}{0.0329}\right)+d$$

$$=d-10.653\left(e^{-0.0329x}+e^{0.0329x}\right)$$

Since $y=192$ when $x=0$,

$192=d-10.653\left(e^{-0.0329(0)}+e^{0.0329(0)}\right)$

$\Rightarrow d=213$ (to 3 s.f.)

Thus, $y=213-10.7\left(e^{-0.0329x}+e^{0.0329x}\right)$ (to 3 s.f.)

OR

Since $y=0$ when $x=96$,

$0=d-10.653\left(e^{-0.0329(96)}+e^{0.0329(96)}\right)$

$\Rightarrow d=251$ (to 3 s.f.)

Thus, $y=251-10.7\left(e^{-0.0329x}+e^{0.0329x}\right)$ (to 3 s.f.)

Common errors:

- Considering $\frac{d^2y}{dx^2}$ (which is pointless for this question part; please integrate part **(ii)** result)
- $e^{0.0329x}=e^{0.0329}\times e^x$ (demonstrating very poor grasp of the laws of indices, a *Secondary Math* concept)
- $e^{-0.0329x}-e^{0.0329x}=2e^{(\text{something})x}$ (thinking that the exponents can somehow be combined even though the powers are unequal)
- $\int e^{0.0329x}\,dx=0.0329e^{0.0329x}+c$ (differentiating instead of integrating)
- forgetting to include the arbitrary constant (which is a lethal mistake for a differential equation problem)
- leaving the final solution with an arbitrary constant (The curve has a fixed position in the x-y plane; thus clearly this question requires a *particular* solution.)
- "Since $y=192$ when $x=0$, d (or c) = 192." (being very flippant and not paying due diligence in evaluating the arbitrary constant, thus failing to realise that $e^0=1$ and not 0)
- not simplifying the final answer

Question 12 (Vectors II)

(i) π_1 : $\mathbf{r}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=25$ and π_2 : $\mathbf{r}\cdot\begin{pmatrix}1\\k\\-2\end{pmatrix}=-15$

Substituting $(x,0,z)$.

$\begin{pmatrix}x\\0\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=5\Rightarrow 2x+z=25$ ...... (1)

$\begin{pmatrix}x\\0\\z\end{pmatrix}\cdot\begin{pmatrix}1\\k\\-2\end{pmatrix}=-3\Rightarrow x-2z=-15$ ...... (2)

Solving (1) and (2), we get $x=7$ and $z=11$.

Thus the position vector of such a point on l_1 is $\begin{pmatrix}7\\0\\11\end{pmatrix}$.

Direction vector of $l_1=\begin{pmatrix}2\\-3\\1\end{pmatrix}\times\begin{pmatrix}1\\k\\-2\end{pmatrix}=\begin{pmatrix}6-k\\1-(-4)\\2k+3\end{pmatrix}=\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}$

Hence a vector equation of l_1 is

$$\mathbf{r}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix},\quad \lambda\in\mathbb{R}$$

__Skills/Concepts Tested__

- Find the equation of the line of intersection between two planes **without** the use of GC by
  - solving two equations with three unknowns through expressing two variables in terms of the third, or
  - finding the point on the line and the direction of the line (i.e. cross product of the normal vectors of the two planes).
- Write an equation of a line

__Common Mistakes__

- Interpreted "independent of k" as letting $k=0$
- Did erroneous workings to find k before proceeding to use GC to find the equation of the line of intersection
- Did not know what to get out from "solving the 2 equations with 3 unknowns" hence the workings appeared aimless
- Careless algebraic manipulation
- Missing "$\mathbf{r}=$" and/or "$\lambda\in\mathbb{R}$" when writing an equation of a line

__Overall Comments__

- Poorly attempted

(i) Alternatively,

$\pi_1 : \mathbf{r}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=5 \Rightarrow 2x-3y+z=25 \cdots\cdots (1)$ and

$\pi_2 : \mathbf{r}\cdot\begin{pmatrix}1\\k\\-2\end{pmatrix}=-3 \Rightarrow x+ky-2z=-15 \cdots\cdots (2)$

Take $y=t$. Then

$(1): 2x-3t+z=25 \Rightarrow z=25+3t-2x \quad (3)$

$(2): x+kt-2z=-15 \quad (4)$

Substituting equation (3) into (4),

$x+kt-2(25+3t-2x)=-15$

$x+kt-50-6t+4x=-15$

$5x+(k-6)t=35$

$x=7+\left(\frac{6-k}{5}\right)t$

Substituting into equation (3),

$z=25+3t-2\left[7+\left(\frac{6-k}{5}\right)t\right]$

$=25+3t-14-\left(\frac{12-2k}{5}\right)t$

$=11+\left(\frac{3+2k}{5}\right)t$

Thus, l_1 has equation

$\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}7\\0\\11\end{pmatrix}+t\begin{pmatrix}\left(\frac{6-k}{5}\right)t\\t\\\left(\frac{3+2k}{5}\right)t\end{pmatrix}, t\in\mathbb{R}$ or

$\mathbf{r}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}, \lambda=\frac{t}{5}\in\mathbb{R}$

(ii) $l_1 : \mathbf{r}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}, \quad \lambda\in\mathbb{R}$

$l_2 : \mathbf{r}=\begin{pmatrix}-15\\0\\0\end{pmatrix}+\beta\begin{pmatrix}3\\1\\-1\end{pmatrix}, \quad \beta\in\mathbb{R}$

Since both lines are skew, they are not parallel and not intersecting.

Suppose both lines are parallel, we have

$\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}=t\begin{pmatrix}3\\1\\-1\end{pmatrix}$. Then $t=5$.

For x-coordinate: $6-k=3(5)\Rightarrow k=-9$

For z-coordinate: $2k+3=-5\Rightarrow k=-4$

Hence we conclude lines are not parallel regardless of k.

Suppose both lines are intersecting, we have

$\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}=\begin{pmatrix}-15\\0\\0\end{pmatrix}+\beta\begin{pmatrix}3\\1\\-1\end{pmatrix}$

Then $\beta=5\lambda$

and $7+\lambda(6-k)=-15+3(5\lambda)\Rightarrow\lambda(k+9)=22$

and $11+\lambda(2k+3)=-5\lambda\Rightarrow\lambda(k+4)=-\frac{11}{2}$

Hence $\frac{k+9}{k+4}=-4\Rightarrow k+9=-4k-16\Rightarrow k=-5$

Therefore for both lines to be skew, k can be any real number except -5.

<u>Skills/Concepts Tested</u>

- Skew lines are **NOT** parallel and **NOT** intersecting
- For such questions, students are expected to assume the lines **ARE** parallel and intersecting to find the value(s) of k before concluding that k **CANNOT** take those values found

<u>Common Mistakes</u>

- Students did not attempt to find the value(s) of k when the lines are parallel. For the few who did, most did not carry out correct workings for this or did not interpret the "solutions" correctly
- When assuming that the 2 lines intersect, students left k in terms of unknowns and did not know that they need to find the value(s) explicitly (which is expected in the question)
- Students attempted to directly "solve" equations like $\begin{pmatrix}\\\\\end{pmatrix}\neq\begin{pmatrix}\\\\\end{pmatrix}$

<u>Overall Comments</u>

- Most students did not attempt this part
- For the few who did, it was very poorly attempted.

(iii)
$\overrightarrow{OA}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}2\\5\\11\end{pmatrix}$ for some $\lambda\in\mathbb{R}$
$\overrightarrow{OB}=\begin{pmatrix}-15\\0\\0\end{pmatrix}+\beta\begin{pmatrix}3\\1\\-1\end{pmatrix}$ for some $\beta\in\mathbb{R}$
Then we have $\overrightarrow{AB}\cdot\begin{pmatrix}2\\5\\11\end{pmatrix}=0$ and $\overrightarrow{AB}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=0$
i.e. $\begin{pmatrix}-22+3\beta-2\lambda\\\beta-5\lambda\\-11-\beta-11\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\5\\11\end{pmatrix}=0\Rightarrow-150\lambda=165\Rightarrow\lambda=-\frac{11}{10}$
$\begin{pmatrix}-22+3\beta-2\lambda\\\beta-5\lambda\\-11-\beta-11\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=0\Rightarrow11\beta=55\Rightarrow\beta=5$
Therefore $\overrightarrow{AB}=\begin{pmatrix}-22+3\beta-2\lambda\\\beta-5\lambda\\-11-\beta-11\lambda\end{pmatrix}=\begin{pmatrix}-4.8\\10.5\\-3.9\end{pmatrix}$.
Hence $\left\lvert\overrightarrow{AB}\right\rvert=\sqrt{(-4.8)^2+(10.5)^2+(-3.9)^2}=\sqrt{\frac{297}{2}}$
	Skills/Concepts Tested
	- Find distance between two skew lines by
	  - identifying two points A (on one line) and B (on the other line) such that A and B are nearest to each other (i.e. $\overrightarrow{AB}$ is perpendicular to both lines). The distance between A and B is the distance between the skew lines, or
	  - identifying a point on each line, say C on one line and D on the other, and finding the length of projection of $\overrightarrow{CD}$ on the direction vector that is perpendicular to both lines with the use of an appropriate formula
	
	Common Mistakes
	- Use of wrong formula/vectors to compute the distance
	- Let $\overrightarrow{AB}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and attempt to solve for the 3 unknowns with only 2 equations instead of expressing $\overrightarrow{AB}$ using the equations of l_1 and l_2 which yields 2 unknowns
	
	Overall Comments
	- Poorly attempted
	- A handful of students were not successful with this part because they did not get the correct answer in **(i)**.
	- It is commendable that there were students who could not get any answer in **(i)** made use of $k=4$ and GC to find the equation of l_1 to make some progress in this part.

(iv) Let the acute angle between l_2 and π_1 be θ. $\sin\theta=\frac{\left\lvert\begin{pmatrix}3\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}\right\rvert}{\sqrt{11}\sqrt{14}}=\frac{2}{\sqrt{154}}$	Skills/Concepts Tested - Use an appropriate formula to find the acute angle between a line and a plane directly Common Mistakes - Many students gave the acute angle instead of the sine of the acute angle hence losing 1 mark Overall Comments - More students could get at least 1 mark for this part - Many students did not include the modulus function in the formula but still get the correct answer since the dot product in the numerator is positive.
(v) Let that shortest distance be d. $\sin\theta=\frac{\sqrt{\frac{297}{2}}}{d}=\frac{2}{\sqrt{154}}$ $\Rightarrow d=75.6$ (to 3 s.f.)	Skills/Concepts Tested - Visualisation and linkage to the earlier parts Common Mistakes - Students used the formula for length of projection to find the shortest distance Overall Comments - Most students did not attempt this part - For those who did, none did this part by using answers from the earlier parts and a simple trigo ratio. Instead they used an onerous method of finding $\overrightarrow{OP}$ followed by a formula to find the shortest distance (which most of these students applied the wrong formula)

Question 1 (Complex Numbers – Cartesian Forms)

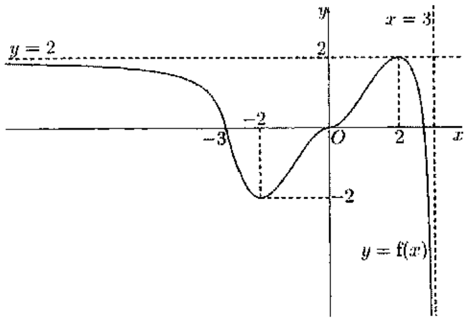
$w^* = z - 2\mathrm{i}$ --- (1) $wz^* = \lvert w\rvert^2 + 6\mathrm{i}$ --- (2) Multiply w throughout (1), we get $ww^* = w(z-2\mathrm{i})$ $\Rightarrow \lvert w\rvert^2 = w(z-2\mathrm{i})$ $\Rightarrow wz^* = w(z-2\mathrm{i}) + 6\mathrm{i}$ (from (2)) $\Rightarrow w(z^* - z + 2\mathrm{i}) = 6\mathrm{i}$ $\Rightarrow w(-2b\mathrm{i} + 2\mathrm{i}) = 6\mathrm{i}$, where $z = a + \mathrm{i}b,\ a,\ b \in \mathbb{R}$ $\Rightarrow w = \dfrac{6\mathrm{i}}{(-2b\mathrm{i}+2\mathrm{i})} = \dfrac{3}{-b+1}$ Hence $w^* = \dfrac{3}{-b+1} = z - 2\mathrm{i} = a + \mathrm{i}b - 2\mathrm{i}$. Comparing the real and imaginary parts, we get $b - 2 = 0 \quad \therefore b = 2$ $\dfrac{3}{-b+1} = a \quad \therefore a = \dfrac{3}{-2+1} = -3$ Therefore, $z = -3 + 2\mathrm{i}$ and $w = \dfrac{3}{-2+1} = -3$. <u>Alternative (but not recommended)</u>: Let $w = a + b\mathrm{i}$ and $z = c + d\mathrm{i}$ where $a, b, c, d \in \mathbb{R}$. From (1): $a - b\mathrm{i} = c + (d-2)\mathrm{i}$ $\Rightarrow a = c$ and $b = 2 - d$. From (2): $a - b\mathrm{i} = c + (d-2)\mathrm{i}$ $\Rightarrow a = c$ --- (3) and $b = 2 - d$ --- (4). $wz^* = \lvert w\rvert^2 + 6\mathrm{i}$ --- (2) $(a + b\mathrm{i})(c - d\mathrm{i}) = a^2 + b^2 + 6\mathrm{i}$ $(ac + bd) + (bc - ad)\mathrm{i} = a^2 + b^2 + 6\mathrm{i}$ $\Rightarrow ac + bd = a^2 + b^2$ ---(5) and $bc - ad = 6$ ---(6). ⋮ (Many steps required to solve (3), (4), (5) and (6)) ⋮	Most students were able to demonstrate the required skills such as conjugation, substitution, and comparing real and imaginary parts. There is a significant number of students who obtained correct answers <u>by chance</u> through incorrect methods and/or inaccuracies. This is because in this particular question, $w = -3$ which is a real number, and real numbers satisfy relations that are generally **INCORRECT** for complex numbers such as $\lvert w\rvert^2 = w^2$ and $w = w^*$. Common mistakes: × Carelessness in computations. × Not stating $a, b \in \mathbb{R}$ when introducing w or z as $a + b\mathrm{i}$. × Treating $\lvert w\rvert^2$ as w^2, or $\lvert a + b\mathrm{i}\rvert^2$ as $a^2 + 2ab\mathrm{i} - b^2$. × $\lvert z - 2\mathrm{i}\rvert^2 = z^2 - 4\mathrm{i}z + (2\mathrm{i})^2$. × $\lvert z - 2\mathrm{i}\rvert^2 = z^2 + 2^2$, or worse still, $\lvert z - 2\mathrm{i}\rvert^2 = 1^2 + 2^2 = 5$. × Taking conjugate of $z - 2\mathrm{i}$ as $z + 2\mathrm{i}$. (It should be $z^* + 2\mathrm{i}$.) × Writing "comparing coefficients" when comparing real parts and imaginary parts. × Writing "comparing constants" when comparing real parts. (Note that complex numbers such as $4 + 5\mathrm{i}$ are constants too.) Students who rewrote both z and w in cartesian forms right from the beginning were not always successful in solving for the 4 unknowns. Hence, this is not a recommended approach, especially since the 4 equations for the 4 unknowns are not all linear.

Question 2 (Arithmetic and Geometric Series)

(i)	$a + (4-1)d = br^{6-1} \Rightarrow a + 3d = br^5$ --- (1) $a + (9-1)d = br^{9-1} \Rightarrow a + 8d = br^8$ --- (2) $a + (11-1)d = br^{14-1} - br^6 \Rightarrow a + 10d = br^{13} - br^6$ --- (3) (2) − (1) : $d = \dfrac{b}{5}(r^8 - r^5)$ (3) − (2) : $d = \dfrac{b}{2}(r^{13} - r^8 - r^6)$ Thus we have $\dfrac{b}{5}(r^8 - r^5) = \dfrac{b}{2}(r^{13} - r^8 - r^5)$ Since $b \neq 0$ and $r \neq 0$ $2r^8 - 2r^5 = 5r^{13} - 5r^8 - 5r^6$ $2r^3 - 2 = 5r^8 - 5r^3 - 5r$ $5r^8 - 7r^3 - 5r + 2 = 0$ Using GC to solve, the only real roots are 1.1371 and 0.34347. Since the geometric series is convergent, $r = 0.3435$ (4 d.p)	Most students could set up the first 2 equations but were unable to use them to show the required expression. Students who could not show the expression were still able to find the correct value of r using GC. Common mistakes: × Do not know which unknown to eliminate. × Aimlessly manipulating the equations without the end in mind.
(ii)	Required sum $= \dfrac{b(0.3435)^{n+1-1}}{1 - 0.3435}$ $\approx 1.523b(0.3435)^n$ $\leq 1.523b(0.3435)^1$ $\approx 0.52315b$ $< \dfrac{3}{5}b$ (shown)	Most students used $S_\infty - S_n$ with the correct substitution of 1st term and common ratio. But majority did not find 0.523. Common mistakes: × Sloppy conclusion. Some simply wrote $0.52315 < \dfrac{3}{5}$.

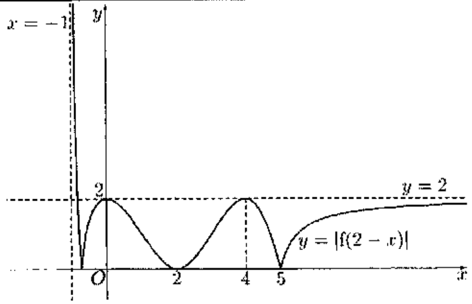
Question 3 (Transformation of Graphs)

(i)



Students gave all variations of answers indicating that they were unable to conclude which regions of the graph was in the negative region.

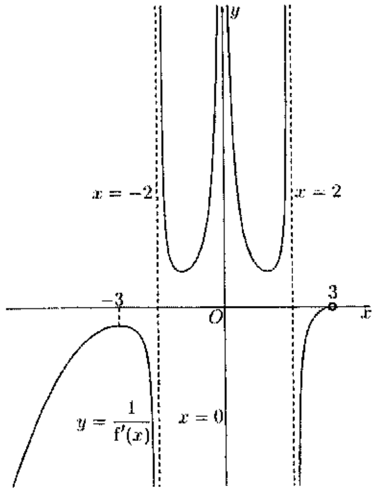
7(ii)



Common mistakes:

× Many carried out a reflection about the y-axis before the translation of 2 units in the negative x direction.

(iii)



Majority are able to do but left out the x-coordinate of the maximum point.

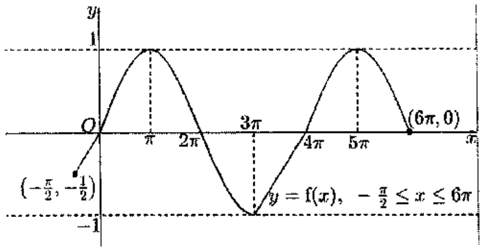
Common mistakes:

× Some continued with the increasing graph in the region $x > 3$.

× Some had their maximum and/or minimum points touching the x-axis.

Question 4 (Applications of Integration)

(a)(i)



Skills/Concepts Tested

- $f(x)=f(x+n\pi)$ for all $x\in\mathbb{R}$ means that f is a periodic function with period of $n\pi$, i.e. the graph repeats itself in every interval of width $n\pi$.

Common Mistakes

- Students drew $y=\frac{x}{\pi}-4$ in the interval $-\frac{\pi}{2}\le x\le 0$ instead of repeating the linear graph in $\frac{7\pi}{2}\le x\le 4\pi$.
- The max/min points and the x-intercepts were not clearly labelled.
- The end-point on the left is incorrect.
- The sine graph in $2\pi\le x\le 3\pi$ appeared more like a line than a curve that is concave upwards.

(a)(ii) $\int_{-\frac{\pi}{2}}^{6\pi}|f(x)|\,dx=\frac{1}{2}\times\frac{\pi}{2}\times\frac{1}{2}+5\int_0^{\pi}\sin\frac{x}{2}\,dx+\frac{1}{2}\times\pi\times 1$

$=\frac{5\pi}{8}+5\left[-2\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$

$=\frac{5\pi}{8}+5(0+2)$

$=\frac{5\pi}{8}+10$

Skills/Concepts Tested

- The definite integral for the region under the x-axis is negative (hence students need to refer to their graph in the earlier part to formulate the integrals accordingly)
- Since periodic graphs "repeat", students need not carry out integration for every interval for $-\frac{\pi}{2}\le x\le 6\pi$.
- Students should only carry out integration of the functions within the defined interval, i.e. $0\le x\le 4\pi$.

Common Mistakes

- $\int_{-\frac{\pi}{2}}^{0}\frac{x}{\pi}-4\,dx$ (should not integrate the function outside of the defined interval)

	- $\int_{2\pi}^{3\pi} \sin\frac{x}{2}\,dx$ (should be $-\int_{2\pi}^{3\pi} \sin\frac{x}{2}\,dx$ since region is below the x-axis) - $\int_{0}^{3\pi} \sin\frac{x}{2}\,dx$ (this is wrong because the graph of $y=\sin\frac{x}{2}$ in $0 \le x \le 3\pi$ cuts the x-axis at $x=2\pi$, i.e. there is a region above the x-axis and another below the x-axis) <u>Overall Comments</u> - Students are given credit so long as their formulated integral is correctly done based on the graph sketched above (which may be wrong) - Students are encouraged to write $-\int_{x_1}^{x_2} f(x)\,dx$ or $\left\|\int_{x_1}^{x_2} f(x)\,dx\right\|$ if the region is under the x-axis rather than $\int_{x_1}^{x_2}\|f(x)\|\,dx$ since we cannot integrate $\|f(x)\|$.

(b) $y=\ln(2x)$ $x=\frac{e^y}{2}$ $x^2=\frac{e^{2y}}{4}$ Since $V_A=V_B$, $\pi\int_0^h \frac{e^{2y}}{4}\,dy=\pi\left(\frac{e}{2}\right)^2(1-h)-\pi\int_h^1\frac{e^{2y}}{4}\,dy$ $\pi\int_0^1 \frac{e^{2y}}{4}\,dy=\frac{\pi e^2}{4}(1-h)$ $\int_0^1 e^{2y}\,dy=e^2(1-h)$ $\left[\frac{e^{2y}}{2}\right]_0^1=e^2(1-h)$ $\frac{e^2-1}{2}=e^2-e^2h$ $e^2h=\frac{e^2+1}{2}$ $h=\frac{e^2+1}{2e^2}$ or $\frac{1}{2}+\frac{1}{2}e^{-2}$ or $\frac{1}{2}+\frac{1}{2e^2}$	<u>Skills/Concepts Tested</u> - We use the volume formula (i.e. $\pi\int_{y_1}^{y_2} x^2\,dy$) only when the region **UNDER** the curve (wrt the y-axis) is rotated about the y-axis. - If the region is **NOT UNDER** the curve, the formula cannot be used directly to obtain the desired volume. In this case, we need to subtract $\pi\int_{y_1}^{y_2} x^2\,dy$ from the volume of cylinder. <u>Common Mistakes</u> - Wrong volume formulation used, e.g. $\int_{y_1}^{y_2} x^2\,dy$, $\pi\int_{y_1}^{y_2} x\,dy$, $\pi\int_{y_1}^{y_2} y^2\,dy$, $\pi\int_{x_1}^{x_2} y^2\,dx$. - For the volume of solid when region B is rotated about the y-axis, students did not subtract $\pi\int_{y_1}^{y_2} x^2\,dy$ from the volume of cylinder. <u>Overall Comments</u> Generally well done by most students (apart from the poor algebraic manipulation to simplify to get the answer)

Question 5 (Integration Techniques)

(a)

$$\int \frac{x}{\left(1+ax^2\right)^2}\,dx$$

$$=\frac{1}{2a}\int \frac{2ax}{\left(1+ax^2\right)^2}\,dx$$

$$=\frac{1}{2a}\frac{\left(1+ax^2\right)^{-1}}{(-1)}+c$$

$$=-\frac{1}{2a\left(1+ax^2\right)}+c$$

$$\int \frac{ax^2}{\left(1+ax^2\right)^2}\,dx$$

$$=\int (ax)\frac{x}{\left(1+ax^2\right)^2}\,dx$$

$$=\frac{-1}{2a\left(1+ax^2\right)}(ax)-\int (a)\frac{-1}{2a\left(1+ax^2\right)}\,dx$$

$$=\frac{-x}{2\left(1+ax^2\right)}+\frac{1}{2}\int \frac{1}{\left(1+ax^2\right)}\,dx$$

$$=\frac{-x}{2\left(1+ax^2\right)}+\frac{1}{2a}\int \frac{1}{\left(\frac{1}{a}+x^2\right)}\,dx$$

$$=\frac{-x}{2\left(1+ax^2\right)}+\frac{1}{2a}\left(\frac{1}{1/\sqrt{a}}\right)\tan^{-1}\left(\frac{x}{1/\sqrt{a}}\right)+c$$

$$=\frac{-x}{2\left(1+ax^2\right)}+\frac{\sqrt{a}}{2a}\tan^{-1}\left(\sqrt{a}x\right)+c$$

This part is generally quite well done; most candidates correctly identified the integral as a variation of the form $\int \frac{f'(x)}{f'(x)}\,dx$ and proceeded by introducing the factor $2a$ in the expression.

Common Mistakes

- Some students reduced the power from -2 to -3 instead of increasing it to -1.

Many students could not apply integration by parts in the next part correctly due to poor algebraic techniques and wrote things like

$$\frac{ax^2}{\left(1+ax^2\right)^2}=\left(\frac{ax}{\left(1+ax^2\right)^2}\right)\left(\frac{x}{\left(1+ax^2\right)^2}\right)$$

Many candidates who managed to do integration by parts could identify that integrating $\frac{1}{1+ax^2}$ gives rise to the inverse tangent function but presented their answers with incorrect factors in their expressions e.g. $\frac{x}{\sqrt{a}}$, $2a$ etc.

(b)

$$x=\frac{1}{y}\ \Rightarrow\ \frac{dx}{dy}=-\frac{1}{y^2}$$

When $x=a\sqrt{2}$, $y=\frac{1}{a\sqrt{2}}$; when $x=2a$, $y=\frac{1}{2a}$

$$\int_{\sqrt{2}a}^{2a}\frac{1}{x\sqrt{x^2-a^2}}\,dx$$

$$=\int_{\frac{1}{\sqrt{2}a}}^{\frac{1}{2a}}\frac{y}{\sqrt{\left(\frac{1}{y}\right)^2-a^2}}\left(-\frac{1}{y^2}\right)dy$$

$$=-\int_{\frac{1}{\sqrt{2}a}}^{\frac{1}{2a}}\frac{1}{y\sqrt{\frac{1-(ay)^2}{y^2}}}\,dy$$

$$=-\int_{\frac{1}{\sqrt{2}a}}^{\frac{1}{2a}}\frac{1}{\sqrt{1-(ay)^2}}\,dy$$

$$=-\frac{1}{a}\int_{\frac{1}{\sqrt{2}a}}^{\frac{1}{2a}}\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2-y^2}}\,dy$$

$$=-\frac{1}{a}\left[\sin^{-1}(ay)\right]_{\frac{1}{\sqrt{2}a}}^{\frac{1}{2a}}$$

$$=-\frac{1}{a}\left[\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$=-\frac{1}{a}\left[\frac{\pi}{6}-\frac{\pi}{4}\right]$$

$$=\frac{\pi}{12a}$$

Most candidates could proceed with the given substitution with some of them having incorrect working for the derivatives.

The change of limits are present in most submissions at the correct step, i.e. when dy is considered in the integral.

A number of students could not simplify $\frac{y}{\sqrt{\left(\frac{1}{y}\right)^2-a^2}}\left(-\frac{1}{y^2}\right)$ to $\frac{1}{\sqrt{1-(ay)^2}}$ and thus did not continue further.

For those who could, most of them correctly identified that the integral involves the inverse sine function; a few other candidates misidentified the integral as the one involving the natural logarithm.

Similar to the previous part, the factor in the integrated expression was often incorrect and thus many candidates could not obtain the correct final expression.

Question 6 (Binomial Distribution)

$A \sim \mathrm{B}(2n+1, p)$ $\mathrm{P}(A=n)=0.009542$ $\binom{2n+1}{n}p^n(1-p)^{n+1}=0.009542$ --- (1) $\mathrm{P}(A=n+1)=0.004090$ $\binom{2n+1}{n+1}p^{n+1}(1-p)^n=0.004090$ --- (2) Therefore, we have $\dfrac{\binom{2n+1}{n+1}p^{n+1}(1-p)^n}{\binom{2n+1}{n}p^n(1-p)^{n+1}}=\dfrac{0.004090}{0.009542}$ $\Rightarrow \dfrac{\frac{(2n+1)!}{n!(n+1)!}}{\frac{(2n+1)!}{(n+1)!n!}}\cdot\dfrac{p^{n+1}}{p^n}\cdot\dfrac{(1-p)^n}{(1-p)^{n+1}}=0.428631 \text{ or } \dfrac{2045}{4771}$ $\Rightarrow \dfrac{p}{1-p}=0.428631 \text{ or } \dfrac{2045}{4771}$ (shown) $\Rightarrow p=0.300$ (to 3 s.f.) Hence $n=17$ (from GC) and $\mathrm{Var}(A)=35(0.300)(0.700)=7.35$ (to 3 s.f.)	Almost the whole cohort could use the probability distribution of a binomial distribution to set up an equation involving p and n, and thus simplified the answer to the required form. Among these, most could also obtain the correct value of n. However, many candidates overlooked that the number of observations in A is $2n+1$ instead of n. Thus many gave their answers as 3.57 which were derived from $np(1-p)$ instead of $(2n+1)p(1-p)$. These candidates usually scored 3 marks.

Question 7 (Sampling Theory)

Assuming n is large, by Central limit Theorem, $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{36}{n}\right)$ approximately. $\mathrm{P}\left(\lvert\overline{X}-\mu\rvert>0.5\right)<0.02$ $\mathrm{P}\left(\left\lvert\dfrac{\overline{X}-\mu}{6/\sqrt{n}}\right\rvert>\dfrac{0.5}{6/\sqrt{n}}\right)<0.02$ $\mathrm{P}\left(\lvert Z\rvert>\dfrac{\sqrt{n}}{12}\right)<0.02$ By symmetry, $\mathrm{P}\left(Z>\dfrac{\sqrt{n}}{12}\right)<0.01$ Using GC $\dfrac{\sqrt{n}}{12}>2.326348$ $\sqrt{n}>27.916$ $n>779.31$ Least n is 780.	About 50% of candidates provided answers such as $n>30$ or $n\geq 30$ which were also accepted. So the part where an assumption is needed was well answered by many. However, there were many who assumed that X has a normal distribution by Central Limit Theorem, which is conceptually wrong. Marks are NOT awarded for those who stated "X is assumed to be normally distributed and hence $\overline{X}$ has a normal distribution" Almost the whole cohort could demonstrate the skill of standardisation. However, many went on to use the table of values to determine the least n instead of using the invNorm function to find the least n (which many also achieved the correct answer and most included 'least').

Question 8 (Discrete Random Variables)

(i)

Dist.	A	B	C	D	E	F
A	-	1		1		
B	-	-	1		1	
C	-	-	-			1
D	-	-	-	-	1	
E	-	-	-	-	-	1
F						

x	1	2	3
$P(X=x)$	$\frac{7}{15}$	$\frac{6}{15}=\frac{2}{5}$	$\frac{2}{15}$

About half the cohort could not get the correct probabilities.

If the question does not state so, do **NOT** assume that there is replacement! So many students thought (A, A), (B, B), … (F, F) were possible cases. Think about this: If you were asked to choose two students from say 23SH01, can it be the same student twice?

Many students approached the questions using a P&C approach. No need. Just list the possible outcomes and **COUNT**.

Order of selection does not matter. However, those who applied order in selection should still get the correct probabilities (as long as they did not assume replacement) since the number of ways without restriction would also be with order accounted for.

(ii)

x	1	2	3
$P(X=x)$	$\frac{7}{15}$	$\frac{6}{15}=\frac{2}{5}$	$\frac{2}{15}$
$\frac{k}{10}x^2$	$\frac{k}{10}$	$\frac{4k}{10}=\frac{2k}{5}$	$\frac{9k}{10}$

Let W denote the player's cash prize, in dollars, in one game.

So, $W=\frac{k}{10}X^2$.

$$E(W)=\frac{k}{10}\left(\frac{7}{15}\right)+\frac{4k}{10}\left(\frac{6}{15}\right)+\frac{9k}{10}\left(\frac{2}{15}\right)$$
$$=\frac{49}{150}k$$

For the stall to be profitable in the long run,

$$E(W)<10$$
$$\frac{49}{150}k<10$$
$$k<\frac{1500}{49}$$
$$=30.612.$$

Therefore, largest value of k is 30.

Students were very careless in reading the question. The question clearly stated that the cash prize was a multiple of the **square** of the player's score. Many students just calculated the expectation of X and multiplied it by $k/10$. Such students capped the maximum credit they could get for this part as 1 mark by their own undoing.

A lot of students thought they could simply multiply the square of E(X) by $k/10$. This is also incorrect as $E(X^2)\neq(E(X))^2$.

Many students proceeded to calculate the variance of X and/or worse, attempted to involve some normal distribution to solve the question. Some of the students who considered the variance of X even showed that they do not even know the correct formula for Var(X). All these observations are very worrying. Please **make sense** of what the question is asking you and do **only** what is necessary.

Question 9 (Hypothesis Testing)

(a)

Step 1: Calculate the unbiased estimates for the population mean and variance.

> **If $y=x-k$, then $\bar{y}=\bar{x}-k\Rightarrow\bar{x}=\bar{y}+k$**

$$\bar{x}=\frac{\sum(x-300)}{60}+300=\frac{-112.8}{60}+300=298.12;$$

> $$s_x^{\,2}=s_y^{\,2}=\frac{1}{n-1}\left[\sum y^2-\frac{\left(\sum y\right)^2}{n}\right]$$

$$s^2=\frac{1}{59}\left(4532.87-\frac{(-112.8)^2}{60}\right)=73.234$$

Please remember the correct formulas for finding unbiased estimates of population parameters.

The unbiased estimates are both terminating decimals in this case, so they should not be rounded off to 3 s.f..

Many students used incorrect symbols to represent the unbiased estimates, such as symbols like $\mu,\sigma^2,\sigma_x^{\,2},\ldots$ Students are encouraged to keep things simple by just writing "$\bar{x}=\ldots$" and "$s^2=\ldots$" instead of giving written phrases because the latter were often incorrect.

Step 2: Define the population mean and state the hypotheses.

Let μ be the population mean volume of filled cans (in ml).

> **To test $H_0:\mu=\mu_0$ against $H_1:\mu\neq\mu_0$**
> **μ = TRUE population mean (unknown)**
> **μ_0 = CLAIMED value of the population mean**

To test $H_0:\mu=300$ against $H_1:\mu\neq300$ at 8% level of significance.

Many students did not define μ. Others did not give complete contextualised definitions, such as merely writing "Let μ be the population mean.", or gave wrong definitions like "sample mean volume".

Many students were evidently confused about how to write the hypotheses. Some wrote "To test $H_0:\mu_0=300$ against $H_1:\mu_1\neq300$" or "To test $H_0:\bar{x}=300$ against $H_1:\bar{x}\neq300$".

(a) [continued] **<u>Step 3: State the distribution of the test statistic.</u>** Under H_0, $Z=\dfrac{\overline{X}-300}{\sqrt{\dfrac{S^2}{60}}}\sim N(0,1)$ approximately by Central Limit Theorem since $n=60$ is large. **<u>Step 4: Calculate the *p*-value and compare it with the level of significance (or calculate the observed test-statistic value and compare it with the critical value).</u>** From GC, p-value $=0.0888>0.08$ **If p-value $\le \alpha$, reject H_0.** **If p-value $> \alpha$, do NOT reject H_0.** (or $\lvert z\rvert=1.702<1.751$). Thus we do not reject H_0.	There were many who wrote $\overline{X}\sim N\left(\boxed{298.12},\dfrac{73.234}{60}\right)$. It should **not** be $\bar{x}$ here! Need to write "approximately" and "by Central Limit Theorem" in this case since distribution of X is unknown. Need to use uppercase letters when writing random variables to be followed by a probability distribution. Cannot write $\dfrac{\bar{x}-300}{\sqrt{\dfrac{s^2}{60}}}\sim N(0,1)$ or $\dfrac{298.12-300}{\sqrt{\dfrac{73.234}{60}}}\sim N(0,1)$ as you would then be implying that a fixed number has a probability distribution (which it does not). There is no need to use normalcdf or invNorm functions to find the p-value or observed test statistic value; just use the TEST function in the GC. Also, do not try to divide p-value or level of significance by 2; these two quantities should include the areas at both "ends" of the normal distribution curve, not just one end. There are students who could not remember which scenario, p-value $\le \alpha$ or p-value $> \alpha$, would lead to H_0 being rejected.

(a) [continued] **<u>Step 5: Write the conclusion in context.</u>** There is insufficient evidence at 8% significance level to **CONCLUDE** (or **CLAIM**) that the mean volume of iced tea per can is not 300 ml.	Some students wrote contradictory statements such as "We do not reject H_0 and conclude there is insufficient evidence to claim that the mean volume of iced tea per can is 300 ml." (the latter part implies insufficient evidence to claim H_0). The conclusion should always be written as "There is sufficient/insufficient evidence to conclude $\mathbf{H_1}$." Thus, you should not write: "We do not reject H_0 and conclude there is sufficient evidence to claim that the mean volume of iced tea per can is 300 ml." (cannot use "accept H_0" phrasing). Also, you should not write affirmatively or assertively like "We do not reject H_0. Thus, the mean volume of iced tea per can is 300 ml." or use assertive words such as "prove", "show". In Hypothesis Testing, we can never make a statement that is *guaranteed* to be true. We are only using sample statistic values to make an inference, to a certain degree of confidence, on the plausibility of the population parameter in question taking on a certain value or certain values.
(b) '8% level of significance' means that there is a probability of 0.08 to <u>conclude</u> (or to <u>claim</u>) that the population mean volume of iced tea per can is not 300 ml when it is actually 300 ml. $\boldsymbol{\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true})}$	To obtain this one mark, • the probability of 0.08 must be written, and as a decimal or a fraction, not as a percentage, • a **clear contextualised** wording for the level of significance must be given. Common mistakes: • "it is the lowest/maximum ..." Once an extremum term is included, no credit will be given. Level of significance refers to the **exact** probability of wrongly rejecting H_0; "lowest..." is for describing p-value. • "probability that the mean volume of iced tea per can is not 300 ml when it is 300 ml". This is referring to $P(H_0 \text{ is not true} \mid H_0 \text{ is true})$, which would always be zero. • "probability that the mean volume of iced tea per can is not 300 ml" missing given condition that H_0 is true (at least must write "wrongly")

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(c) Let Y be the volume of iced tea in a randomly chosen bottle in ml and μ_Y be its population mean.

To test $H_0 : \mu_Y = 500$ against $H_1 : \mu_Y < 500$ at 4 % level of significance

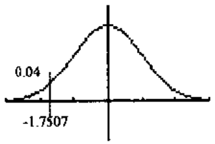
Under H_0, $\bar{Y} \sim N\left(500, \frac{5^2}{35}\right)$.

Observed test statistic $z = \dfrac{\bar{y} - 500}{\left(\frac{5}{\sqrt{35}}\right)}$

Therefore, the critical region for this test is given by

$$\frac{\bar{y} - 500}{\left(\frac{5}{\sqrt{35}}\right)} \le -1.7507$$

$$\Rightarrow 0 \le \bar{y} \le 498.5$$



When a new test is involved, always check to see if the tail has changed. Do not assume that the hypotheses and tail remain the same as the previous part(s)!

To see why this part involves a *lower-tailed* test,

- Firstly, one must understand that under any circumstances, the null hypothesis must be an **equality** and the alternative hypothesis must be a **non-equality** or **inequality**.
- Next, if this was instead an upper-tailed test, **both** rejection and non-rejection of the null hypothesis would favour the claim that the mean volume of iced tea per bottle is indeed *at least* 500ml, making it a very pointless test.

Thus, the only way out is to use a lower-tailed test.

"Critical region **for this test**" means the question is asking: "In the context of *this test*, what are the possible values of the sample mean volume per bottle that would lead to rejection of the null hypothesis?", so giving an inequality for the observed test-statistic value (aka "$z \le \ldots$") is not going to garner any credit.

Population standard deviation (5) is given, so that should be used instead of any unbiased estimate (which in any case, you should not be using the value of s^2 from part **(i)** since that is for volume per *can*). Also, interpret the question carefully - 5 is population standard deviation, **NOT** sample standard deviation! Don't go and write things like $\frac{35}{34}(5^2)$!

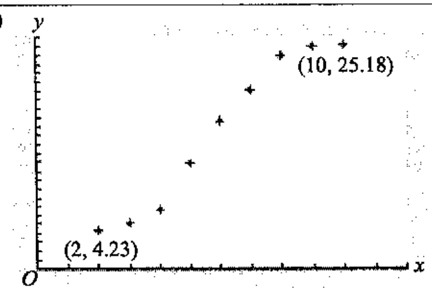
"approximately", "Central Limit Theorem" should not be used since the distribution of Y is **exactly** normal (and population variance is known).

Note that for rejection of the null hypothesis, critical value is included in the critical region, so the inequality should be non-strict.

As the name "critical **REGION**" implies, it refers to a continuous frame like an interval of values, not a discrete number, so please do not write things like "critical region = …" Make sense of what you are writing please.

Question 10 (Correlation and Regression)

(a)



Well done.

Common mistakes:
- ✗ Not labelling the origin.
- ✗ Not labelling the extreme points/values.
- ✗ Drawing 10 points instead of 9.

(b)(i) $r = 0.9567$ — Well done.

(b)(ii) $r = 0.9477$ — Well done.

(c) Since the (absolute) r value for Model C is closer to 1 compared to that of Model D, there is stronger linear correlation between y and $\ln x$ than between y and x^2. Thus, Model C is a better fit.

By GC,
$y = 15.57978 \ln x - 10.70594$ (7 s.f.)
$y = 15.6 \ln x - 10.7$ (3 s.f.)

The explanation was generally well done. However, a significant number of students completely missed the requirement that they had to find the equation of the regression line.

Common mistakes:
- ✗ Final answer not rounded to 3 s.f.
- ✗ $y = 15.6x - 10.7$.
- ✗ Leaving final answer as 26.65 without stating the number of subscriptions as 266.5 million (or 2.665×10^8).
- ✗ Leaving the final answer as $y = 266.5$ million. (The y-value is 26.65)
- ✗ Using "≈" instead of "=" for equation of line.

(d) $y = 15.57978 \ln(11) - 10.70594 = 26.65$.
The number of subscribers in 2024 is 266.5 million.

Since $x = 11$ is not within the given data range of $2 \le x \le 10$, the estimate is not reliable.

Common mistakes:
- ✗ Leaving final answer as 26.65 without stating the number of subscriptions as 266.5 million (or 2.665×10^8).
- ✗ Leaving the final answer as $y = 266.5$ million. (The y-value is 26.65)
- ✗ Using the 3 s.f. version of the equation to calculate y when $x = 11$. This would result in rounding errors.
- ✗ The answer omitted $x = 11$ or the year 2024.
- ✗ The answer omitted the data range for x, [2,10] (or 2015 to 2023).
- ✗ Stating that the estimated y-value of 26.65 is not within the data range as the reason for unreliability. (Whether the estimated y-value is within the data range of y is **irrelevant** in determining reliability. What matters here is only whether $x = 11$ is in the data range of x, namely [2,10], or not.)

(e)	Model C is not suitable in the long run because the model predicts that the number of subscriptions will tend to infinity in the long run, which is not realistic.	Any reasoning along the lines of "y increases as x increases" is insufficient as this description does not exclude the possibility of asymptotic behaviour along a horizontal asymptote. Common mistakes: × Did not include the notion of number of subscriptions <u>tending to infinity</u> or increasing <u>without limit</u>. × Stating that that $\ln x$ increases and tends to a limit (or plateaus) as x increases. (Recall from O-Levels that that $\ln x \to \infty$ as $x \to \infty$.) × Using absolute terms such as "the number of subscribers <u>will/would</u> plateau". × Not stating whether the model is suitable or not.

Question 11(a) (Permutations and Combinations)

(i)	A | B | C | (blank) / (blank row)
	
	Step 1: allocate the group of 3 into the positions such that they are next to one another.
	No. of ways $= 4\times 3!$
	Step 2: Allocate the rest of the 5 students to their positions.
	No. of ways $= 5!$
	
	Total no. of ways such that not all the students in the same class are standing next to one another
	$= 8! - (4\times 3!5!)$
	$= 37440$
	
	Alternatively,
	<u>Case (1)</u> The 3 students are allocated to the same row.
	
	(blank) | X | Y | (blank) / (blank row)
	
	Step 1: Choose one student from the remaining 5 to take position X or Y and allocate the 3 students to take up the remaining position.
	No. of ways
	$= \binom{5}{1}\binom{2}{1}3!$
	$= 60$
	Step 2: Allocate the remaining 4 students in the other rows. No. of ways $= 4!$
	Since there are two different rows, no. of ways for case (1)
	$= 60\times 4!\times 2$
	$= 2880$
	
	<u>Case (2)</u> 2 Students from the same class stand in one row, while the remaining student stand in the other row.
	Step 1: For the row with 2 students, fill up the spaces with 2 more students from the 5 and arrange.
	No. of ways $= \binom{3}{2}\binom{5}{2}4! = 720$
	Step 2: Arrange the students in the other row.
	No. of ways $= 4! = 24$
		<u>Skills/Concepts Tested</u>
		- "Complement" method or to consider cases
		- Read "not all the students in the same class are…" as "not [all the students in the same class are…]" hence we can subtract the number of ways in which [all the students in the same class are…] from the total number of ways to arrange the 8 people
		
		<u>Common Mistakes</u>
		- Actions that need to be taken to achieve the outcome are not complete, e.g.
		  - students tended to forget to choose the people before arranging them
		  - students only arranged people in one row but not the other
		- Total number of ways to arrange the 8 people in the 2 rows is $\binom{8}{4}4!\binom{4}{4}4!\times 2$. The $\times 2$ is redundant and meaningless
		- Students considered micro-cases (which were time-consuming and distracted them from considering cases that cover all grounds) like
		  - 2 students from the same class are in the front row and 2 students from the same class are in the back row
		  - 2 students from the same class in 1 row are separated and 2 students from the same class in 1 row are together
		- For the case where 3 students from the same class are in 1 row, some students forgot to make sure that the 4th person in that row must not be on either end
		
		<u>Overall Comments</u>
		A standard question that is not well attempted by students

	Since there are two different rows, no. of ways for case (2) $= 720\times24\times2$ $= 34560$ Hence total number of ways $= 2880+34560$ $= 37440$	
(ii)	Step 1 : Choose 2 girls and 2 boys to form one row and alternate either GBGB or BGBG No. of ways $=\binom{4}{2}\binom{4}{2}2!2!\times2=288$ Step 2: Repeat alternate for the other row. No. of ways $=\binom{2}{2}\binom{2}{2}2!2!\times2=8$ Total number of ways $=288\times8$ $=2304$ ***Alternatively,*** Step 1. Four cases for boys and girls to alternate : BGBG BGBG GBGB GBGB BGBG , GBGB , BGBG , GBGB Step 2. Slot in the girls and slot in the boys. No. of ways $=4!\times4!$ Total number of ways $=4!\times4!\times4$ $=2304$	Skills/Concepts Tested - "Boys and girls alternate" is **NOT** the same as "boys or girls are separated". Common Mistakes - Actions that need to be taken to achieve the outcome are not complete. - Students did the "slotting-in" method with the intention to separate the boys or the girls. - Students forgot to consider the case where a boy/girl starts first in each row - There is some confusion between the use of the Addition and Multiplication principles among a significant minority Overall Comments - A standard question that is not well attempted by students - Students need to be mindful that even though 2!, 2 and $\binom{2}{1}$ are equal, they convey different meanings. Hence it is important for students to **write the numbers in the correct form to convey the correct intention** to the marker. Correct numerical answers without logical and sound workings will not be given credit.

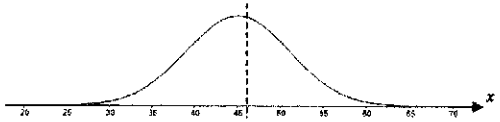
Question 11(b) (Probability)

(i) Observe that all 3 students from Piety have to be selected in order for the conditions to be fulfilled. Case 1: 2 Grace, 1 Joy and 1 Hope ${}^6C_2\times{}^5C_1\times{}^4C_1=300$ Case 2: 2 Joy, 1 Grace and 1 Hope ${}^5C_2\times{}^6C_1\times{}^4C_1=240$ Case 3: 2 Hope, 1 Grace and 1 Joy ${}^4C_2\times{}^6C_1\times{}^5C_1=180$ Total ways = 300 + 240 + 180 = 720 Total ways without restriction $={}^{18}C_7=31824$ Required probability $=\frac{720}{31824}=\frac{5}{221}$ or 0.0226	Well performed by many, scoring 3 or 4 marks. For those who scored 3 marks, it was often due to being careless in their calculations to get the numerical answers However, there were some students who used methods that were not commonly thought and they did not include their explanations. In such cases, full credit was **not** given unless students explained their thinking process.
(ii) Case 1: Teacher is between 2 team leaders from Grace ${}^3C_2\times2!\times(6-1)!=720$ Alternatively, $\binom{3}{2}\times(5-1)!\times5\times2!$ Case 2: Teacher is between 2 team leaders from Hope ${}^2C_2\times2!\times(6-1)!=240$ Alternatively $(5-1)!\times5\times2!$ Total ways without restriction $=7!=5040$ Required Probability $=\frac{720+240}{5040}=\frac{4}{21}$ or 0.190	Well performed by many, scoring full marks. However, there were some students who used methods that were not commonly thought and they did not include their explanations. In such cases, full credit was **not** given unless students explained their thinking process.

Question 12 (Normal Distribution)

(a) Let X denote the random variable 'time taken (min) to install an electricity meter.'

$X \sim N(45, 6^2)$



Note: $P(20 < X < 70) = 0.99997$

Common errors:

- Many roughly drew a bell-shaped curve with symmetry at x = 45, while some did not notice that 45 is the mid-point value between 20 and 70.
- A few wrote 25 instead of 20, or 75 instead of 70.
- Many drew the curves, showing a significant deviation of the curve from the x-axis at both x = 20 and x = 70.

Learning points:

Note that $P(20 < X < 70) = 0.99997$, thus the normal distribution curve should appear "flat" (asymptotic) significantly before reaching the end points. In fact, for any normal distribution, the probability density function should be approximately zero more than 3 standard deviations from the mean.

(b) Let E be the random variable 'number of 'inefficient' houses out of n houses.'

$E \sim B(n, P(X > 60))$

$E \sim B(n, 0.0062097)$

$P(E < 3) = P(E \le 2) \ge 0.90$

Using GC,

n	$P(E \le 2)$
177	0.9012 ≥ 0.9
178	0.8999 < 0.9
179	0.8986

Largest n = 177

Common errors:

- could not identify binomial distribution from the question; found the distribution of sample mean $\overline{X}$ instead
- did not define random variable for a binomial distribution or defined it wrongly.
- did not use the result $P(E < 3) = P(E \le 2)$; some used complement method.
- Some used MF26 formula for Binomial probability distribution function and could not proceed on to find largest n.
- a few did not write "largest n" in the final answer.

Learning points:

- $E \sim B(n, 0.0062097)$, probability of success is $P(X > 60)$; should use at least **TWO** s.f or d.p. than final answer for intermediate steps
- Define binomial random variable E as number of <u>successes</u> out of <u>n trials</u> in context
- Can use $Y_1 = \text{binomcdf}(x, 0.0062097, 2)$ in GC; do not need to use binomial pdf formula in MF26.
- For solving inequalities involving integer-valued variables, should show table of 2 rows of values is using table in GC; if using graph in GC, should write down the answer in equality, then write down "least/greatest n ="

(c) Let W be the random variable 'amount of electricity (kWh) used in the household in a month.'

$W \sim \text{N}(524,\ 27^2)$

Let $T = 0.26(W_1 + W_2)$.

$\text{E}(T) = 0.26 \times 524 \times 2 = 272.48$

$\text{Var}(T) = 0.26^2 \times 2 \times 27^2 = 98.5608$

$T \sim \text{N}(272.48,\ 98.5608)$

$$\text{P}(T > 270 \mid 250 < T < 280) = \frac{\text{P}(270 < T < 280)}{\text{P}(250 < T < 280)} = \frac{0.36425}{0.76383} = 0.490 \quad (3\text{ s.f.})$$

We need to assume that the electricity used in each of these two months is independent for this particular household.

Common errors:

- Mistook $W_1 + W_2$ as $2W$
- Did not apply $\text{Var}(aX) = a^2\text{Var}(X)$ correctly.
- Used standard deviation instead of variance; used 27 instead of 27^2 as the variance
- did not apply conditional probability correctly. The numerator should be $\text{P}(A \cap B)$. Some students wrote $\text{P}(A)\text{P}(B)$. This is wrong as we do not know if A and B are independent. Must take intersection of the two events $T > 270$ and $250 < T < 280$
- For assumption needed, errors include "independence across households", "amount of electricity used is the same for each month", sample size > 30", "probability of success is the same", "random selection of household"…

Learning points:

- Be familiar with the laws of expectation and variance for independent random variables. Clear working should be shown to secure method marks in case of arithmetic errors.
- You are advised to express as a combination of independent random variable first, for e.g. $T = 0.26(W_1 + W_2)$, before finding the corresponding expectation and variance. Remember to write ~N(mean, variance)
- Only need to define W.
- Important: $W_1 + W_2 \neq 2W$.

(d) Let G denote the random variable 'time taken (min) to install a gas meter

$G \sim \text{N}(\mu, \sigma^2)$.

$\text{P}(G < 38) = 0.15, \qquad \text{P}(G > 53) = 0.4$

$\text{P}\left(Z < \frac{38-\mu}{\sigma}\right) = 0.15, \qquad \text{P}\left(Z > \frac{53-\mu}{\sigma}\right) = 0.4$

By G.C.,

$\frac{38-\mu}{\sigma} = -1.0364, \qquad \frac{53-\mu}{\sigma} = 0.25335$

$\mu - 1.0364\sigma = 38$ ---(1), $\mu + 0.25335\sigma = 53$ ---(2)

Solving (1) and (2):

$\mu = 50.0535 = 50.1 \quad (3\text{ s.f.})$

$\sigma = 11.6298 = 11.6 \quad (3\text{ s.f.})$

Most students can answer this part well. It is advised to translate the given information into two simultaneous equations and solve them using their calculator. Take note to write the equation in the form $ax + by = c$.

Common errors:

- wrote $\frac{\mu - X}{\sigma^2}$, $\frac{X-\mu}{\sigma}$
- did not use InvNorm to get the z-value, still wrote as 0.4, 0.15 on the RHS of the equation.