# RVHS 2024 H2 Math Prelim P1

1 The curve $C$ has equation $y = x^3 + x$. It undergoes the transformations in the following order:

Translation by 2 units in the negative $y$-direction, followed by

scaling parallel to the $x$-axis with scale factor $\frac{1}{2}$, followed by

reflection about the $x$-axis.

**(a)** Determine the equation of the resulting curve. [4]

**(b)** Find the coordinates of the point of intersection between the two curves. [1]

**2** **(a)** Solve the inequality $\frac{3x-4x^2}{2x+1} \geq 0$ by algebraic method. [3]

**(b)** Hence solve the inequality $\frac{4|x|^2-3|x|}{2|x|+1} \leq 0$. [3]

**3** It is given that $y=(1+x)^x$.

**(a)** By considering $\ln y$, find $\dfrac{dy}{dx}$ in terms of $x$. [4]

**(b)** Find $\dfrac{dw}{dx}$ in terms of $x$ if $w=(1+x)^x+(1+2x)^{2x}$. [3]

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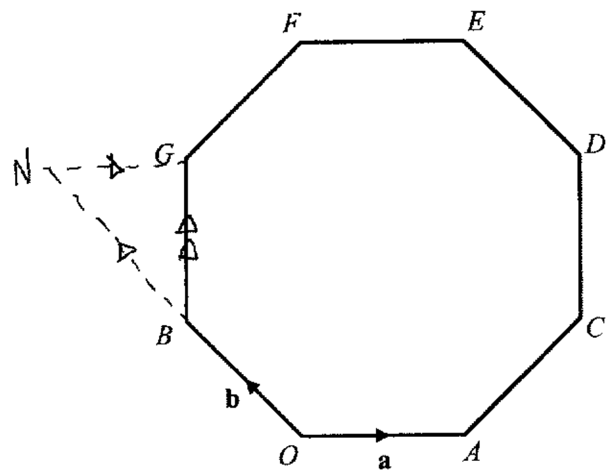



The origin $O$ and regular octagon $OACDEFGB$ lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ (see diagram).

**(a)** Explain why $\overrightarrow{BG}$ can be expressed as $\overrightarrow{BG} = s\mathbf{a} + t\mathbf{b}$ for real constants $s$ and $t$. [2]

It is given that angle $AOB =$ angle $OBG = 135°$.

**(b)** It is known that line $BG$ is perpendicular to line $OA$. By considering the scalar product $\overrightarrow{BG} \cdot \overrightarrow{OA}$, show that $t = \sqrt{2}s$. [3]

**(c)** By considering a suitable scalar product, or otherwise, deduce the values of $s$ and $t$. [3]

**5** **Do not use a calculator in answering this question.**

**(a)** The complex number $z$ is given by

$$z=\frac{\left(-\sqrt{3}-\mathrm{i}\right)^5}{\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)}.$$

Find $|z|$ and $\arg(z)$. [4]

**(b)** **(i)** The roots of the equation $w^2=4\mathrm{i}$ are $w_1$ and $w_2$. Find $w_1$ and $w_2$ in cartesian form $x+\mathrm{i}y$, showing your working. [3]

**(ii)** Hence, or otherwise, find in exact cartesian form the roots $v_1$ and $v_2$ of the equation

$$v^2-10v+(25-\mathrm{i})=0.$$ [3]

6 **(a)** Show that $\ln\left(2^{r-1}\sin 2\theta\right)-\ln\left(2^{r}\sin\theta\right)=\ln\left(\cos\theta\right)$. [2]

**(b)** By letting $\theta=\frac{1}{2^{r}}$, find $\sum_{r=1}^{n}\ln\left(\cos\left(\frac{1}{2^{r}}\right)\right)$ in terms of $n$. [3]

**(c)** Hence, show that $\sum_{r=1}^{\infty}\ln\left(\cos\left(\frac{1}{2^{r}}\right)\right)$ converges and state its value.

(You may assume that $2^{n}\sin\left(\frac{1}{2^{n}}\right)\to 1$ as $n\to\infty$.) [2]

**7** **(a)** Given that $y^3 = \mathrm{e}^{ax}\cos x$, where $a$ is a constant, show that $3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - ay^3 = -\mathrm{e}^{ax}\sin x$. [2]

**(b)** By further differentiation of this result, find the Maclaurin series for $y$, up to and including the term in $x^2$. [5]

**(c)** Given that the first three non-zero terms in the above Maclaurin series are equal to the first three non-zero terms in the series expansion of $\mathrm{e}^{x+bx^2}$, where $b$ is a constant, find the values of $a$ and $b$. [3]

**8** There are two identical tanks, each of capacity 90 000 $m^3$. Robots A and B are each programmed to fill up an empty tank with water at the end of each day.

Robot A fills the tank with 6000 $m^3$ of water on the first day. For each subsequent day, Robot A fills the tank with 50 $m^3$ of water lesser than the previous day.

Robot B fills the tank with 9000 $m^3$ of water on the first day. For each subsequent day, Robot B fills the tank with 85% of the volume of water it fills the tank in the previous day.

**(a)** Find the number of days for robot A to fill up the tank. [3]

**(b)** Determine with clear reasoning whether robot B would be able to fill up the tank with water. [1]

**(c)** Find the total amount of water that robot B fills in the tank by the end of the $10^{th}$ day. [2]

**(d)** At the start of the $11^{th}$ day, robot B is reprogrammed. At the end of the $11^{th}$ day, it fills the tank with 5% more volume of water it fills on the previous day and continues to do so for each subsequent day.
Show that the total volume of water, in $m^3$, that Robot B fills in the tank after reprogramming can be expressed as

$$189000(0.85)^9\left(1.05^n-1\right),$$

where $n$ is the number of days starting from the $11^{th}$ day.
Hence, determine with clear reasoning which robot will be faster in filling up the tank with the above change. [5]

**9** The closed curve $C$, which is symmetrical about the line $x = 0$, has parametric equations

$$x = \cos 3t + \cos t, \qquad y = -2\cos 2t,$$

for $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

**(a)** Sketch $C$. [1]

**(b)** Find the exact equation of the tangent of $C$ at the point when $t = \frac{\pi}{4}$. [3]

**(c)** Find the acute angle between the two tangents of curve $C$ at $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$. [2]

**(d)** Show that the area enclosed by the curve $C$ is given by

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 3\sin 5t + \sin 3t + 2\sin t \, dt.$$

Hence find the area enclosed by the curve $C$ correct to 3 decimal places. [4]

**10** In a robotics competition, toy cars move along straight lines to complete tasks. Points are defined relative to the origin $(0,0,0)$. The $x$-, $y$- and $z$-axes are in the directions east, north and vertically upwards respectively, with units in centimetres.

The position vectors of two toy cars $A$ and $B$, with respect to time $t$ in seconds, are given as $\mathbf{r}_\mathbf{A} = t(5\mathbf{i}+\mathbf{k})$ and $\mathbf{r}_\mathbf{B} = \mathbf{i}+6\mathbf{j}-\mathbf{k}+t(4\mathbf{i}-\mathbf{j}+\mathbf{k})$ respectively.

**(a)** Show that after two seconds, car $B$ is at the point with coordinates $(9,4,1)$ and find the distance that car $A$ has travelled in the same duration. [2]

**(b)** Determine whether cars $A$ and $B$ meet. [3]

**(c)** Explain why cars $A$ and $B$ travel on a common plane surface and show that the cartesian equation of the surface is $x-y-5z=0$. [5]

A drone flies above the cars to capture images of the cars during the competition. The shortest distance between the drone and the surface where cars $A$ and $B$ travel is maintained at 50 cm.

**(d)** Find the cartesian equation of the plane containing the flight path of the drone. [2]

**11** An experiment was conducted at room temperature, where the levels of the concentration of a chemical is investigated over time.
The initial concentration of the chemical was $x_0$ mol/dm$^3$. A possible model suggests that the rate at which the concentration decreases is directly proportional to $x^2$, where $x$ mol/dm$^3$ is the concentration of the chemical at time $t$ minutes after the start of the experiment.

**(a)** **(i)** By setting up and solving a differential equation, show that the time taken for the concentration of the chemical to reach $\frac{x_0}{2}$ is inversely proportional to $x_0$. [4]

**(ii)** It was observed that it took 4 min and 16 min to reach one-half and one-quarter of $x_0$ respectively. Explain why the above model is not suitable. [2]

For the rest of the question, take $x_0 = \frac{3}{2}$.

It was later discovered that the concentration of the chemical can be modelled in an alternative way. Due to a reversible reaction, the rate at which the concentration of the chemical increases is directly proportional to $\left(\frac{3}{2}-x\right)^2$ while the rate at which it decreases is directly proportional to $x^2$.

It is given that there is no change in the concentration when the concentration is $\frac{1}{2}$ mol/dm$^3$.

**(b)** **(i)** For this model, show that $\frac{dx}{dt} = -k\left(4x^2+4x-3\right)$, where $k$ is a positive real constant. [3]

**(ii)** Solve this differential equation to find $x$ in terms of $t$ and $k$. [5]

# RVHS 2024 H2 Math Prelim P2

**Section A: Pure Mathematics [40 marks]**

**1** **(a)** Find $\int \ln x \, dx$. [2]

**(b)** The region $A$ is bounded by the curve $y = \frac{1}{2}\sqrt{\ln x}$, $x$-axis and the line $x = 5$. Find the exact volume when $A$ is rotated $2\pi$ radians about the $x$-axis. [3]

**2** Functions f and g are defined by

$$f : x \mapsto x + e^x, \text{ for } x \in \mathbb{R}, x > -2,$$

$$g : x \mapsto \ln x \text{ for } x \in \mathbb{R}, x > \frac{1}{e}.$$

**(a)** Show that f has an inverse. [1]

**(b)** Show that the composite function fg exist and find $\mathrm{fg}(x)$. [3]

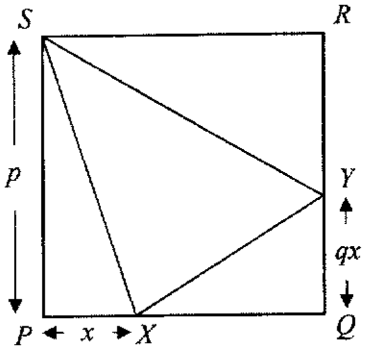
**(c)** Hence find the value of $x$ which satisfies $\mathrm{g}(x) = \mathrm{f}^{-1}(x)$. [3]

**3** A function f is defined by $\mathrm{f}(x)=ax+b+\frac{c}{x-1}$, where $a$, $b$ and $c$ are constants. The graph of $y=\mathrm{f}(x)$ has a minimum point at $(2.5,13)$ and also passes through the $y$-axis at $(0,-12)$.

**(a)** Find the values of $a$, $b$ and $c$. [4]

**(b)** Sketch the graph of $y=\mathrm{f}(x)$, stating clearly any asymptotes, axial intercepts and turning points. [3]

**4**




The diagram shows a square $PQRS$ of side $p$ metres. The points $X$ and $Y$ lie on $PQ$ and $QR$ respectively such that $PX = x$ m and $QY = qx$ m, where $q$ is a constant such that $q > 1$.

**(a)** Given that the area of triangle $XYS$ is $A$ m$^2$, show that $A = \frac{1}{2}\left(qx^2 - px + p^2\right)$. [3]

**(b)** Given that $x$ can vary, show that $QY = YR$ when $A$ is minimum and express the minimum value of $A$ in terms of $p$ and $q$. [6]

**5** **Do not use a calculator in answering this question.**

Let $\mathrm{P}(z)$ be a polynomial with real coefficients and $z = re^{i\theta}$ is one the roots of the equation $\mathrm{P}(z) = 0$.

**(a)** Show that $z^2 - (2r\cos\theta)z + r^2$ is a quadratic factor of $\mathrm{P}(z)$. [3]

**(b)** Given that $z^4 - z^3 + z^2 - z + 1 = (z^2 - az + 1)(z^2 - bz + 1)$, for real values $a$ and $b$, and that $b < 0 < a$, find exact values for $a$ and $b$. [4]

**(c)** By considering $1 + z^5$, verify that $z = e^{i\frac{\pi}{5}}$ is a root to the equation $z^4 - z^3 + z^2 - z + 1 = 0$. [2]

**(d)** Show that $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$. [3]

## Section B: Statistics [60 marks]

**6** The number 396900 can be expressed as $2^2 \times 3^4 \times 5^2 \times 7^2$.
A factor of 396900 can be expressed in the form $2^a \times 3^b \times 5^c \times 7^d$ for non-negative integers $a$, $b$, $c$ and $d$. For example, 3150 and 140 are factors of 396900 and they can be expressed as $2^1 \times 3^2 \times 5^2 \times 7^1$ and $2^2 \times 3^0 \times 5 \times 7$ respectively.

**(a)** State all the possible values for $a$. [1]

**(b)** Find the number of factors of 396900. [1]

A factor of 396900 is chosen randomly.

**(c)** Find the probability that the chosen factor is divisible by 3 given that it is even. [3]

7 Company A consists of 15 men and 10 women on its staff. 10 staff are to be selected to join a contest as Team A. The random variable $R$ denotes the number of men in Team A.

**(a)** Show that

$$\mathrm{E}(R)=\sum_{r=0}^{10}\left[r\times\frac{\binom{15}{r}\binom{m}{10-r}}{n}\right], \text{ for } r=0, 1, 2, \ldots, 10,$$

where $m$ and $n$ are real constants to be determined. [3]

**(b)** Use your calculator to find E($R$) and $\mathrm{Var}(R)$. [3]

Company B also consists of male and female staff where 10 staff are to be selected to join the same contest as Team B, and $Q$ denotes the number of men in the team.
It is known that E($Q$) = 6.25 and Var($Q$) = 3.

Before both companies finalise their teams, they decide to study the different team configurations further by generating a list of random samples of the teams.

**(c)** Estimate the probability the mean number of men in 30 random samples of Team A exceeds the mean number of men in 30 random samples of Team B. [4]

**8** Long standing data indicates that customers of 80% of all table reservations at a restaurant will turn up. For this question, assume that all tables can only be reserved once in a dinner service and customers stay until the end of dinner service.

**(a)** One day, a restaurant has 14 table reservations for dinner service.

**(i)** Find the expectation and variance for the number of table reservations where the customers turn up. [2]

**(ii)** Find the probability that at least 9 table reservations but less than 13 have their customers turn up. [3]

**(b)** Find the probability that, for dinner service on two days with 14 table reservations each, there is a total of exactly 24 table reservations where the customers turn up. [2]

**(c)** A restaurant manager of a 30-table restaurant decides to offer more table reservations than the full capacity. Find the maximum number of table reservations that the manager can offer such that there is a probability of at least 85% that the restaurant will not exceed full capacity for a dinner service. [3]

9 **(a)** A scientist is studying the growth of water lilies in a large lake. He planted a water lily at one corner and he measures the area, $A$ km$^2$, the water lilies cover on day $t$. His results are recorded below:

| Time, $t$ (days) | 1 | 4 | 7 | 14 | 20 | 29 |
|---|---|---|---|---|---|---|
| Area, $A$ (km$^2$) | 0.6 | 1.3 | 3.7 | 7.4 | 8.6 | 9.2 |

**(i)** Draw a scatter diagram to illustrate the data. [1]

**(ii)** The scientist would like to predict the future growth of the water lilies. Using the scatter diagram and the context of the question, state two reasons why, in this context, a linear model is not appropriate. [2]

**(b)** It is proposed to fit the above data with a model of the form $\ln(D-A) = a + bt$, where $D$ is a suitable constant. The product moment correlation coefficient between $t$ and $\ln(D-A)$ is denoted by $r$. The following table gives values of $r$ for some possible values of $D$.

| $D$ | 9.5 | 9.8 | 10 |
|---|---|---|---|
| $r$ | | −0.99359 | −0.99114 |

**(i)** Calculate the value of $r$ for $D = 9.5$, giving your answer correct to 5 decimal places. Hence, explain which of 9.5, 9.8 or 10 is the most appropriate value of $D$ for the model to fit. [2]

**(ii)** Using this value of $D$, calculate the values of $a$ and $b$ correct to 5 decimal places, and use them to predict the area covered by water lilies after 28 days. Comment on whether the estimate is reliable. [4]

**(iii)** Give an interpretation, in context, of the value of $D$. [1]

**10** An internet advertising company *TicTakAim* claims that viral video's duration has a mean duration of 30 seconds. An influencer wants to investigate the company's claim as she believes that the company is underestimating the mean duration. However, she is unable to record the durations of all the viral videos.

**(a)** Explain how she could obtain a sample of viral video durations, and why she should obtain the sample in this way. [2]

The influencer takes a sample of 90 viral video's durations. The viral video's durations, $x$ seconds, are summarised as follows.

$$\Sigma(x-30)=90 \qquad \Sigma(x-30)^2=2037$$

**(b)** Find the unbiased estimates of the population mean and variance of the durations of viral videos. [2]

**(c)** Carry out an appropriate test, at the 3% level of significance, whether the company's claim is justifiable. You should state your hypotheses and define any symbols you use. [4]

**(d)** Explain, in the context of the question, the meaning of "at the 3% level of significance". [1]

**(e)** The influencer was later informed that the population standard deviation of the viral video duration is $\sigma$ seconds. Find the set of values of $\sigma$ so that the influencer can conclude that there is sufficient evidence at the 3% level of significance to believe that *TicTakAim* is underestimating the mean duration. [3]

**11** A leather craftsman customized leather belts according to the widths of the customer's buckles. Over a period of time, it is found that the buckle widths are normally distributed. 60% of the buckles have width more than 25 mm and 15% are less than 24 mm.

**(a)** Find the mean and variance of the buckle width. [3]

The widths of the leather belts produced by the craftsman follow a normal distribution with mean 25.1 mm and standard deviation 1.4 mm.

**(b)** Find the probability that the width of a randomly chosen leather belt is between 24 mm and 26 mm. [1]

In order to fit the leather belts nicely into the buckles, the craftsman reduces the widths of these leather belts by 1%.

**(c)** Find the probability that the total width of 3 randomly chosen leather belts is less than 75.4 mm. [3]

There are holes that are punctured into the leather belts that have diameters, in mm, that follow the distribution $N(4.5, 0.2^2)$.

The prong is part of the belt buckle that is also known as the pin or the "fork". It goes through any of the holes in the belt to secure the belt in place.

The diameter of the prong, in mm, follows the distribution $N(4.3, 0.1^2)$.

If the diameter of a prong is more than 0.2 mm greater than the diameter of a hole, then the hole has to be enlarged to make it fit.

If the diameter of a hole is more than 0.3 mm greater than the diameter of a prong, welding is done to increase the diameter of the prong to make it fit.

**(d)** A complete set of a belt is made up of a randomly chosen buckle with a prong and a leather belt with 5 punctured holes. Find the probability that for a belt, the prong can be fitted into every hole without having the holes enlarged or the prong welded. [4]

**(e)** A punctured hole on a belt and a buckle with a prong are randomly chosen for inspection. State with a reason whether or not the event that the hole needs to be enlarged and the event that the prong needs to be welded are independent. [2]

# Solution and Comments for 2024 H2 Math Prelim P1

1 The curve $C$ has equation $y = x^3 + x$. It undergoes the transformations in the following order:

Translation by 2 units in the negative $y$-direction, followed by

scaling parallel to the $x$-axis with scale factor $\frac{1}{2}$, followed by

reflection about the $x$-axis.

**(a)** Determine the equation of the resulting curve. [4]

**(b)** Find the coordinates of the point of intersection between the two curves. [1]

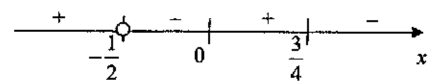
| 1 | Solutions [5] Graphing Transformations. | Comments |
|---|---|---|
| (a) | $y = x^3 + x$<br>$\downarrow$ A : $y \to y+2$<br>$y = x^3 + x - 2$<br>$\downarrow$ B : $x \to 2x$<br>$y = 8x^3 + 2x - 2$<br>$\downarrow$ C : $y \to -y$<br>$y = -8x^3 - 2x + 2$<br>The equation of the resulting curve is $y = -8x^3 - 2x + 2$. | Many students were still unable to obtain the full credit for this part due to slips in A: replacing $y$ by $y-2$ and C: replacing $x$ by $-x$ which are incorrect.<br>(Note: Should students coincidentally get the correct answer but steps in method are incorrect, they will not be awarded any answer mark!) |
| (b) | Solving $x^3 + x = -8x^3 - 2x + 2$,<br>$x = 0.429303 \Rightarrow y = 0.508424$<br>$\therefore$ coordinates of intersection point is $(0.429, 0.508)$. | Students who got (a) correct would usually get (b) correct. |





2 **(a)** Solve the inequality $\frac{3x-4x^2}{2x+1} \geq 0$ by algebraic method. [3]

**(b)** Hence solve the inequality $\frac{4|x|^2 - 3|x|}{2|x|+1} \leq 0$. [3]

| 2 | Solution [6] Inequalities | Comments |
|---|---|---|
| (a) | For solving $\frac{3x-4x^2}{2x+1} \geq 0$,<br>we first have $\frac{x(3-4x)}{2x+1} \geq 0$.<br>Then using number line testing method:<br><br>Thus the solutions are:<br>$x < -\frac{1}{2}$ or $0 \leq x \leq \frac{3}{4}$. | While there are few methods in solving this question, students ought to learn what is more efficient in their work so that they can minimise the time spent on the question.<br>There were quite group of students who did long division but could not proceed on. Some common mistakes are:<br>• Wrong calculation of the sign for the number line<br>• Wrong choice of region after multiplying by "-1"<br>• Simply forgetting that $x \neq -\frac{1}{2}$ |
| (b) | Next for solving $\frac{4\|x\|^2 - 3\|x\|}{2\|x\|+1} \leq 0$, we note that<br>$\frac{4\|x\|^2 - 3\|x\|}{2\|x\|+1} \leq 0 \Rightarrow \frac{3\|x\| - 4\|x\|^2}{2\|x\|+1} \geq 0$.<br>Thus, we can replace $x$ with $\|x\|$ in the first given inequality $\frac{3x-4x^2}{2x+1} \geq 0$.<br>Hence the solution to the 2nd inequality should be:<br>$\|x\| < -\frac{1}{2}$ **(No solution as $\|x\| \geq 0$)** or $0 \leq \|x\| \leq \frac{3}{4}$<br>So, the solution is $-\frac{3}{4} \leq x \leq \frac{3}{4}$. | Some students saw the replacement of $x$ in part (a) with $\|x\|$ but they did not test out the required region and assumed that part (b) wants the other region as required in part (a).<br>Students must also check on the final answer with several regions in the number line, for example, $0 \leq x \leq \frac{3}{4}$ or $-\frac{3}{4} \leq x \leq 0$ would imply $-\frac{3}{4} \leq x \leq \frac{3}{4}$.<br>A lot of students simply reject $\|x\| < -\frac{1}{2}$ without any |



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| | | |
|---|---|---|
| | | explanation which is not accepted. Furthermore, some students $\lvert x\rvert < -\frac{1}{2} \Rightarrow x \in \mathbb{R}$ |





**3** It is given that $y=(1+x)^x$.

**(a)** By considering $\ln y$, find $\frac{dy}{dx}$ in terms of $x$. [4]

**(b)** Find $\frac{dw}{dx}$ in terms of $x$ if $w=(1+x)^x+(1+2x)^{2x}$. [3]

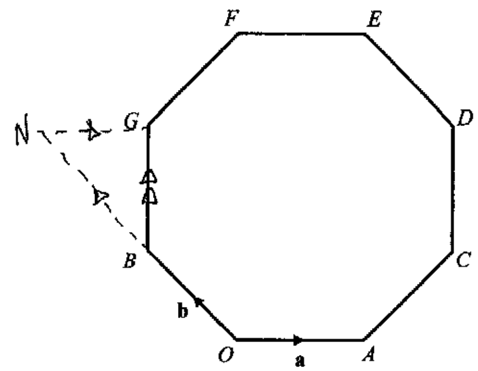
| 3 | Solutions [7] Differentiation Techniques | Comments |
|---|---|---|
| (a) | $y=(1+x)^x$<br>$\ln y = x\ln(1+x)$<br>Differentiate both sides wrt $x$:<br>$\frac{1}{y}\frac{dy}{dx}=\frac{x}{1+x}+\ln(1+x)$<br>$\frac{dy}{dx}=y\left[\frac{x}{1+x}+\ln(1+x)\right]$<br>$=(1+x)^x\left[\frac{x}{1+x}+\ln(1+x)\right]$<br>$=x(1+x)^{x-1}+(1+x)^x\ln(1+x)$ | Generally well done.<br><br>Except a minority of students who incorrectly included modulus within their workings for (a) and (b) i.e. $\ln y = x\ln\lvert(1+x)\rvert$ which was unnecessary in this case. |
| (b) | Consider $v=(1+2x)^{2x}$.<br>From part (i) and chain rule,<br>$\frac{dv}{dx}=\left[2x(1+2x)^{2x-1}+(1+2x)^{2x}\ln(1+2x)\right]\frac{d(2x)}{dx}$<br>$=2(1+2x)^{2x}\left[\frac{2x}{1+2x}+\ln(1+2x)\right]$<br>$=4x(1+2x)^{2x-1}+2(1+2x)^{2x}\ln(1+2x)$.<br><br>$\therefore$ For $w=(1+x)^x+(1+2x)^{2x}$,<br>$\frac{dw}{dx}=x(1+x)^{x-1}+(1+x)^x\ln(1+x)$<br>$+4x(1+2x)^{2x-1}+2(1+2x)^{2x}\ln(1+2x)$.<br><br>**<u>Alternative</u>**<br>Let $v=(1+2x)^{2x}$<br>$v=(1+2x)^{2x}$<br>$\ln v = 2x\ln(1+2x)$<br>Differentiate both sides wrt $x$: | There were many students who did not succeed in this part due to a few common errors:<br><br><u>First common error:</u><br>$w=(1+x)^x+(1+2x)^{2x}$<br>implying<br>$\ln w = x\ln(1+x)+2x\ln(1+2x)$<br>is **<u>NOT TRUE!!</u>**<br>Note that<br>$\ln(A+B)\neq \ln A+\ln B$<br><br><u>Second common error:</u><br>$\frac{dw}{dx}=x(1+x)^{x-1}+2x(1+2x)^{2x-1}$<br>is **<u>NOT TRUE!!</u>**<br>Note that<br>$\frac{d}{dx}(x^n)=nx^{n-1}$ only if $n$ is a constant but here $x$ is not a constant!<br><br><u>Third common error:</u><br>When differentiating $(1+2x)^{2x}$, besides replacing $x$ |

| | | |
|---|---|---|
| | $\frac{1}{v}\frac{dv}{dx}=\frac{4x}{1+2x}+2\ln(1+2x)$<br>$\frac{dv}{dx}=v\left[\frac{4x}{1+2x}+2\ln(1+2x)\right]$<br>$=(1+2x)^{2x}\left[\frac{4x}{1+2x}+2\ln(1+2x)\right]$<br><br>$\frac{dw}{dx}=x(1+x)^{x-1}+(1+x)^{x}\ln(1+x)$<br>$+(1+2x)^{2x}\left[\frac{4x}{1+2x}+2\ln(1+2x)\right].$ | by $2x$ in (a)'s result, by chain rule there is also a need to differentiate $2x$ i.e.<br>$2(1+2x)^{2x}\left[\frac{2x}{1+2x}+\ln(1+2x)\right]$<br>Many students missed out the "2" in their final answer. |





**4**




The origin $O$ and regular octagon $OACDEFGB$ lie in the same plane, where $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$ (see diagram).

**(a)** Explain why $\overrightarrow{BG}$ can be expressed as $\overrightarrow{BG}=s\mathbf{a}+t\mathbf{b}$ for real constants $s$ and $t$. [2]

It is given that angle $AOB=$ angle $OBG=135°$.

**(b)** It is known that line $BG$ is perpendicular to line $OA$. By considering the scalar product $\overrightarrow{BG}\cdot\overrightarrow{OA}$, show that $t=\sqrt{2}s$. [3]

**(c)** By considering a suitable scalar product, or otherwise, deduce the values of $s$ and $t$. [3]

| 4 | Solution [8] Abstract Vectors | Comments |
|---|---|---|
| (a) | Let the plane that contains the pentagon be $\pi$.<br>Hence, any point on the plane has the form<br>$\mathbf{r}=\mathbf{b}+s\mathbf{a}+t\mathbf{b}$, for some real parameters $s$ and $t$.<br>In particular,<br>$\overrightarrow{OG}=\mathbf{b}+s\mathbf{a}+t\mathbf{b}$ for some $s,t\in\mathbb{R}$.<br>$\overrightarrow{OG}=\overrightarrow{OB}+s\mathbf{a}+t\mathbf{b}$<br>$\overrightarrow{OG}-\overrightarrow{OB}=s\mathbf{a}+t\mathbf{b}$<br>$\overrightarrow{BG}=s\mathbf{a}+t\mathbf{b}$, for real constants $s,t$. (Shown)<br><br><u>Alternate Solution</u><br>Define a point N such that NBO are collinear and NG is parallel to OA (see annotated diagram),<br>$\overrightarrow{BN}=t\mathbf{b}$<br>$\overrightarrow{NG}=s\mathbf{a}$, for real constants $s,t$.<br>Hence $\overrightarrow{BG}=s\mathbf{a}+t\mathbf{b}$, for real constants $s,t$. (Shown) | Not well done.<br>The intent of this question is for you to form an equation of plane that contains the polygon. SO one should start from the definition of equation of plane that has the form $\mathbf{r}=\mathbf{b}+s\mathbf{a}+t\mathbf{b}$, for some real parameters $s$ and $t$.<br><br>Some *out-of-the-box (literally, as one has to define another point out of the octagon)* solutions were presented and it seems an easier approach using just basic vectors by applying triangle law of addition. |



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| | | |
|---|---|---|
| (b) | $\overrightarrow{BG}\cdot\overrightarrow{OA}=0$<br>$(s\mathbf{a}+t\mathbf{b})\cdot(\mathbf{a})=0$<br>$s\mathbf{a}\cdot\mathbf{a}+t\mathbf{b}\cdot\mathbf{a}=0$<br>$s\lvert\mathbf{a}\rvert^2+t\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos\angle AOB=0$<br>$s\lvert\mathbf{a}\rvert^2+t\lvert\mathbf{a}\rvert\lvert\mathbf{a}\rvert\cos135°=0$<br>$s-\frac{t}{\sqrt{2}}=0$<br>$t=\sqrt{2}s$ (Shown) | Most can start with fact that the dot product of the 2 vectors is 0.<br><br>Most showed the expansion clearly but failed to recognize that the length of OB and OA are equal ($\lvert\mathbf{b}\rvert=\lvert\mathbf{a}\rvert$) since they are sides of a regular octagon.<br>Also students are reminded to write the dot (.) clearly. |
| (c) | $\overrightarrow{BO}\cdot\overrightarrow{BG}=\lvert\overrightarrow{BO}\rvert\lvert\overrightarrow{BG}\rvert\cos\angle OBG$<br>$(-\mathbf{b})\cdot(s\mathbf{a}+t\mathbf{b})=\lvert\mathbf{b}\rvert^2\cos135°$<br>$-s\mathbf{b}\cdot\mathbf{a}-t\lvert\mathbf{b}\rvert^2=-\frac{\lvert\mathbf{b}\rvert^2}{\sqrt{2}}$<br>$-s\lvert\mathbf{b}\rvert^2\cos135°-t\lvert\mathbf{b}\rvert^2=-\frac{\lvert\mathbf{b}\rvert^2}{\sqrt{2}}$<br>$\frac{s}{\sqrt{2}}-t=-\frac{1}{\sqrt{2}}$<br>$s-\sqrt{2}t=-1$<br><br>Sub $t=\sqrt{2}s$ into $s-\sqrt{2}t=-1$.<br>$s-2s=-1\Rightarrow s=1$ and therefore, $t=\sqrt{2}$.<br><br>**<u>Alternatively</u>**<br>$\overrightarrow{BG}\cdot\overrightarrow{BG}=\lvert\overrightarrow{BG}\rvert^2$<br>$(s\mathbf{a}+t\mathbf{b})\cdot(s\mathbf{a}+t\mathbf{b})=\lvert\mathbf{a}\rvert^2$<br>$s^2\lvert\mathbf{a}\rvert^2+2st(\mathbf{a}\cdot\mathbf{b})+t^2\lvert\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2$<br>$s^2\lvert\mathbf{a}\rvert^2+2st(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos135°)+t^2\lvert\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2$<br>$s^2-\sqrt{2}st+t^2=1$<br><br>Sub $t=\sqrt{2}s$ into $s^2-\sqrt{2}st+t^2=1$.<br>$s^2-2s^2+2s^2=1\Rightarrow s=\pm1$ and therefore,<br>If $s=1, t=\sqrt{2}$.<br>If $s=-1, t=-\sqrt{2}$ it is reject as coefficient of **b** has got positive. | Many students attempted this very carelessly. Take note that we are required to consider a suitable scalar product and it is very unlikely that it will be the same dot product we did at part (b).<br><br>In considering $\overrightarrow{BO}\cdot\overrightarrow{BG}$, note that the angle they make with each other is 135° and not 45°. However if it is $\overrightarrow{OB}\cdot\overrightarrow{BG}$, then yes it is 45°.<br>Draw diagrams (with arrows to see) to know which angles we are referring to (arrows should be diverging out).<br><br>Some rather common mistakes include squaring vectors, which is illegal and not allowed! We can only square magnitude of vector (modulus). |





**5 Do not use a calculator in answering this question.**

**(a)** The complex number $z$ is given by

$$z=\frac{\left(-\sqrt{3}-\mathrm{i}\right)^5}{\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)}.$$ [4]

Find $|z|$ and $\arg(z)$.

**(b)** **(i)** The roots of the equation $w^2=4\mathrm{i}$ are $w_1$ and $w_2$. Find $w_1$ and $w_2$ in cartesian form $x+\mathrm{i}y$, showing your working. [3]

**(ii)** Hence, or otherwise, find in exact cartesian form the roots $v_1$ and $v_2$ of the equation

$$v^2-10v+(25-\mathrm{i})=0.$$ [3]

| 5 | Solution [10] Complex Numbers | Comments |
|---|---|---|
| (a) | Now,<br>$\lvert-\sqrt{3}-\mathrm{i}\rvert=\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}=2$<br>and $\arg\left(-\sqrt{3}-\mathrm{i}\right)=-\left(\pi-\frac{\pi}{6}\right)=-\frac{5\pi}{6}$.<br>$\lvert\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)\rvert=1$<br>and $\arg\left(\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)\right)=-\frac{\pi}{7}$<br><br>Hence,<br>$\lvert z\rvert=\frac{\lvert-\sqrt{3}-\mathrm{i}\rvert^5}{\lvert\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)\rvert}=\frac{2^5}{1}=32.$<br>And,<br>$\arg(z)=5\arg\left(-\sqrt{3}-\mathrm{i}\right)-\arg\left(\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)\right)$<br>$=5\left(-\frac{5\pi}{6}\right)-\left(-\frac{\pi}{7}\right)$<br>$=-\frac{169\pi}{42}$<br>$=-\frac{\pi}{42}$ (principle range) | Some students simply ignore the instructions at the start of the question and used GC to provide the answer. **If there is insufficient working to justify a correct answer, NO marks is awarded.**<br><br>One common mistake is that the $\arg(z)=-\frac{169\pi}{42}$ and students did not provide the final answer in principle range.<br><br>Quite a lot of students also provided the $\arg\left(\cos\left(\frac{1}{7}\pi\right)-\mathrm{i}\sin\left(\frac{1}{7}\pi\right)\right)$ as $\frac{\pi}{7}$ instead and hence leading to wrong answer.<br>Also, quite a lot of students provided $\arg\left(-\sqrt{3}-\mathrm{i}\right)=\frac{\pi}{6}$ as they did not visualise the quadrant where $-\sqrt{3}-\mathrm{i}$ is located and provided the wrong angle |
| (b) | $(x+\mathrm{i}y)^2=4\mathrm{i}$<br>$x^2+2xy\mathrm{i}-y^2=4\mathrm{i}$<br>$x^2-y^2+2xy\mathrm{i}=0+4\mathrm{i}$ | This question is badly attempted and some simply used the GC to provide the correct answer which result in NO marks awarded. |

| | | |
|---|---|---|
| | By comparing parts;<br>Re: $x^2-y^2=0 \Rightarrow x=\pm y$<br>Im: $2xy=4 \Rightarrow y=\dfrac{2}{x}$<br><br>If $y=-x$,<br>$-x=\dfrac{2}{x} \Rightarrow -x^2=2$ (No solutions as $x^2 \geq 0$ for real $x$.)<br><br>If $y=x$,<br>$x=\dfrac{2}{x} \Rightarrow x=\pm\sqrt{2} \Rightarrow y=\pm\sqrt{2}$.<br><br>Hence the possible numbers are,<br>$\sqrt{2}+\sqrt{2}\mathrm{i}$ or $-\sqrt{2}-\sqrt{2}\mathrm{i}$. | Quite a few students used the quadratic formula and resulted in the same answer and cannot move on.<br><br>Note: When expressing a complex number is expressed in exponential form, $w=re^{i\theta}$ and $r \geq 0$. |
| (c) | Method 1:<br>$v^2-10v+(25-\mathrm{i})=0$<br>$v=\dfrac{10\pm\sqrt{(-10)^2-4(25-\mathrm{i})}}{2}$<br>$v=\dfrac{10\pm\sqrt{4\mathrm{i}}}{2}$<br>$v=\dfrac{10\pm(\sqrt{2}+\sqrt{2}\mathrm{i})}{2}$<br>$v=5+\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\mathrm{i}$ or $5-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\mathrm{i}$<br><br>Method 2:<br>$v^2-10v+(25-\mathrm{i})=0$<br>$v^2-10v+25=\mathrm{i}$<br>$(v-5)^2=\mathrm{i}$<br>$(2v-10)^2=4\mathrm{i}$<br>$2v-10=\sqrt{2}+\sqrt{2}\mathrm{i}$ or $-\sqrt{2}-\sqrt{2}\mathrm{i}$<br>$v=5+\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\mathrm{i}$ or $5-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\mathrm{i}$ | This part of the question is badly attempted.<br><br>Method 1 is the easier and students can see quite clearly the use of part (b) here.<br><br>For students using Method 2, there are slips in some of their work. |





6 (a) Show that $\ln(2^{r-1}\sin 2\theta)-\ln(2^r\sin\theta)=\ln(\cos\theta)$. [2]

(b) By letting $\theta=\dfrac{1}{2^r}$, find $\sum_{r=1}^{n}\ln\left(\cos\left(\dfrac{1}{2^r}\right)\right)$ in terms of $n$. [3]

(c) Hence, show that $\sum_{r=1}^{\infty}\ln\left(\cos\left(\dfrac{1}{2^r}\right)\right)$ converges and state its value.

(You may assume that $2^n\sin\left(\dfrac{1}{2^n}\right)\to 1$ as $n\to\infty$.) [2]

| 6 | Solutions [7] MoD | Comments |
|---|---|---|
| (a) | $\ln(2^{r-1}\sin 2\theta)-\ln(2^r\sin\theta)$<br>$=\ln\left(\dfrac{2^{r-1}\sin 2\theta}{2^r\sin\theta}\right)$<br>$=\ln\left(\dfrac{2^r\sin\theta\cos\theta}{2^r\sin\theta}\right)$<br>$=\ln(\cos\theta)$ (Shown) | A handful of students used a wrong log law $\ln\left(\dfrac{a}{b}\right)=\dfrac{\ln a}{\ln b}$. |
| (b) | Let $\theta=\dfrac{1}{2^r}$,<br>$\ln(2^{r-1}\sin 2\theta)-\ln(2^r\sin\theta)=\ln(\cos\theta)$<br>$\ln\left(2^{r-1}\sin 2\left(\dfrac{1}{2^r}\right)\right)-\ln\left(2^r\sin\left(\dfrac{1}{2^r}\right)\right)=\ln\left(\cos\left(\dfrac{1}{2^r}\right)\right)$<br>$\ln\left(2^{r-1}\sin\left(\dfrac{1}{2^{r-1}}\right)\right)-\ln\left(2^r\sin\left(\dfrac{1}{2^r}\right)\right)=\ln\left(\cos\left(\dfrac{1}{2^r}\right)\right)$<br><br>$\sum_{r=1}^{n}\ln\left(\cos\left(\dfrac{1}{2^r}\right)\right)$<br>$=\sum_{r=1}^{n}\ln\left(2^{r-1}\sin\left(\dfrac{1}{2^{r-1}}\right)\right)-\ln\left(2^r\sin\left(\dfrac{1}{2^r}\right)\right)$<br>$=\ln(\sin 1)-\ln\left(2\sin\left(\dfrac{1}{2}\right)\right)$<br>$+\ln\left(2\sin\left(\dfrac{1}{2}\right)\right)-\ln\left(2^2\sin\left(\dfrac{1}{2^2}\right)\right)$<br>$+\ldots$<br>$+\ln\left(2^{n-2}\sin\left(\dfrac{1}{2^{n-2}}\right)\right)-\ln\left(2^{n-1}\sin\left(\dfrac{1}{2^n}\right)\right)$<br>$+\ln\left(2^{n-1}\sin\left(\dfrac{1}{2^{n-1}}\right)\right)-\ln\left(2^n\sin\left(\dfrac{1}{2^n}\right)\right)$<br>$=\ln(\sin 1)-\ln\left(2^n\sin\dfrac{1}{2^n}\right)$ | Majority understood that part (a) needed to be used here. However, instead of substituting $\theta=\dfrac{1}{2^r}$ in part (a) to convert θ to r, many expanded the summation with θ there. Thus, the cancellation of the sine term was not successful. |
| (c) | As $n\to\infty$, $\ln\left(2^n\sin\left(\dfrac{1}{2^n}\right)\right)\to\ln 1$, hence, | Majority had some ideas of using limits and the hint here. However, the |



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| | | |
|---|---|---|
| | $\sum_{r=1}^{n}\ln\left(\cos\frac{1}{2^r}\right)\to\ln(\sin 1)-\ln 1=\ln(\sin 1)$ a finite value.<br>This implies, $\sum_{r=0}^{\infty}\ln\left(\cos\frac{1}{2^r}\right)$ converges.<br><br>In particular,<br>$\sum_{r=0}^{\infty}\ln\left(\cos\frac{1}{2^r}\right)=\lim_{n\to\infty}\sum_{r=0}^{n}\ln\left(\cos\frac{1}{2^r}\right)=\ln(\sin 1)$. | presentation was not satisfactory.<br>E.g. not many used the notation " $\lim_{n\to\infty}$ "; and some wrote<br>$\ln\left(2^{\infty}\sin\left(\frac{1}{2^{\infty}}\right)\right)\to\ln 1$.<br>Many simply evaluated the infinite sum to give the final answer without attempting to explain or show that the infinite sum is convergent. There were actually 2 parts to part (c). |





7 **(a)** Given that $y^3=e^{ax}\cos x$, where $a$ is a constant, show that $3y^2\frac{dy}{dx}-ay^3=-e^{ax}\sin x$. [2]

**(b)** By further differentiation of this result, find the Maclaurin series for $y$, up to and including the term in $x^2$. [5]

**(c)** Given that the first three non-zero terms in the above Maclaurin series are equal to the first three non-zero terms in the series expansion of $e^{x+bx^2}$, where $b$ is a constant, find the values of $a$ and $b$. [3]

| 7 | Solutions [10] Maclaurin Series | Comments |
|---|---|---|
| (a) | $y^3=e^{ax}\cos x$<br>$3y^2\frac{dy}{dx}=ae^{ax}\cos x-e^{ax}\sin x$<br>$=ay^3-e^{ax}\sin x$<br>$\therefore 3y^2\frac{dy}{dx}-ay^3=-e^{ax}\sin x$ (shown) | No issue here. |
| (b) | Differentiate both sides wrt to $x$:<br>$3y^2\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2-3ay^2\frac{dy}{dx}=-ae^{ax}\sin x-e^{ax}\cos x$<br><br>When $x=0$: $y=1$<br>$\frac{dy}{dx}=\frac{a}{3}$<br>$\frac{d^2y}{dx^2}=\frac{a^2-3}{9}$<br><br>The Maclaurin series for $y$<br>$=1+x\left(\frac{a}{3}\right)+\frac{x^2}{2}\left(\frac{a^2-3}{9}\right)+\ldots$<br>$=1+\frac{a}{3}x+\frac{a^2-3}{18}x^2+\ldots$ | Many had difficulties in performing this second round of differentiation, particularly in differentiating $3y^2\frac{dy}{dx}$. By product rule, we should obtain $3y^2\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}\left(6y\frac{dy}{dx}\right)$.<br>When this part was not done correctly, $\left.\frac{d^2y}{dx^2}\right\vert_{x=0}$ was incorrect. |
| (c) | $e^{x+bx^2}=1+\left(x+bx^2\right)+\frac{\left(x+bx^2\right)^2}{2}+\ldots$<br>$=1+x+bx^2+\frac{x^2}{2}+\ldots$<br>$=1+x+\frac{2b+1}{2}x^2+\ldots$<br>$\therefore$ by comparing coefficients,<br>$\frac{a}{3}=1\Rightarrow a=3$ and<br>$\frac{a^2-3}{18}=\frac{2b+1}{2}\Rightarrow b=-\frac{1}{6}$ | A number of students derive the series of $e^{x+bx^2}$ via differentiation instead of using the standard series of $e^x$ in MF26. Some errors were made and led to $x^2$ term.<br><br>A very common mistake in the comparison was that students compared term instead of coefficients. Eg compared |

| | | |
|---|---|---|
| | | $x+bx^2$ with $\frac{a}{3}x$ which is incorrect. |

**8** There are two identical tanks, each of capacity 90 000 m$^3$. Robots A and B are each programmed to fill up an empty tank with water at the end of each day.

Robot A fills the tank with 6000 m$^3$ of water on the first day. For each subsequent day, Robot A fills the tank with 50 m$^3$ of water lesser than the previous day.

Robot B fills the tank with 9000 m$^3$ of water on the first day. For each subsequent day, Robot B fills the tank with 85% of the volume of water it fills the tank in the previous day.

**(a)** Find the number of days for robot A to fill up the tank. [3]
**(b)** Determine with clear reasoning whether robot B would be able to fill up the tank with water. [1]
**(c)** Find the total amount of water that robot B fills in the tank by the end of the 10$^{th}$ day. [2]
**(d)** At the start of the 11$^{th}$ day, robot B is reprogrammed. At the end of the 11$^{th}$ day, it fills the tank with 5% more volume of water it fills on the previous day and continues to do so for each subsequent day.
Show that the total volume of water, in m$^3$, that Robot B fills in the tank after reprogramming can be expressed as

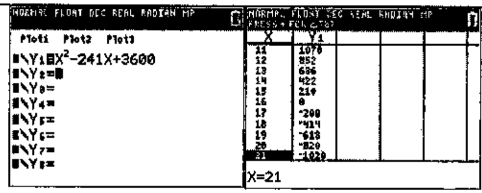
$$189000(0.85)^9\left(1.05^n-1\right),$$

where $n$ is the number of days starting from the 11$^{th}$ day.
Hence, determine with clear reasoning which robot will be faster in filling up the tank with the above change. [5]

| 8 | Solutions [11] AP GP | Comments |
|---|---|---|
| (a) | $a=6000,\ d=-50$<br>$S_n \geq 90000$<br>$\frac{n}{2}[2(6000)+(n-1)(-50)]\geq 90000$<br>$n[12000-50n+50]\geq 180000$<br>$n[241-n]\geq 3600$<br>$n^2-241n+3600\leq 0$<br>$n^2-241n+3600\leq 0$<br>$16\leq n\leq 225$<br><br>Robot A takes 16 days to fill up the tank.<br><br>**Alternative Method**<br>$\frac{n}{2}[2(6000)+(n-1)(-50)]\geq 90000$<br>$n^2-241n+3600\leq 0$ | Generally well done. |








| $n$ | $n^2-241n+3600$ |
|---|---|
| 15 | $210>0$ |
| 16 | 0 |

Robot A takes 16 days to fill up the tank.

| | | |
|---|---|---|
| (b) | $a=9000,\ r=0.85$<br>$S_\infty=\frac{9000}{1-0.85}=60000<90000$<br>Robot B would not be able to fill up the tank as the theoretical maximum volume it would fill is 60000 m$^3$ which is less than the capacity of the tank. | Many students managed to get this part correct.<br><br>There was a minority of students who solved $S_n=90000$ and obtained a negative value for $n$ and concluded from there. However, such an explanation is incomplete and does not demonstrate complete understanding of the context of filling up the tank. They are advised to consider when $n\to\infty$ instead. |
| (c) | $a=9000,\ r=0.85$<br>$S_{10}=\frac{9000\left(1-0.85^{10}\right)}{1-0.85}$<br>$S_{10}=48187.53574$<br>Total amount of water that robot B fills in the tank by the end of the 10$^{th}$ day $\approx 48200$ m$^3$ (3.s.f) | Generally well done.<br><br>Only a very small minority made the mistake of finding $S_9$ instead. |
| (d) | $S_{10}=48187.53574$<br>Let $t_n$ be the volume of water Robot B fills on the $n$th day, starting from the 11$^{th}$ day.<br>Volume of water filled on the 10$^{th}$ day$=9000(0.85)^9$<br>For the 11$^{th}$ day, $t_1=9000(0.85)^9(1.05)$<br>Hence,<br>$t_1+t_2+...+t_n=\frac{9000(0.85)^9(1.05)(1.05^n-1)}{1.05-1}$<br>$=189000(0.85)^9(1.05^n-1)$ (Shown) | This part was poorly done by many students.<br><br>Many failed to show the result as they missed out 1.05 in their working. Note that here, the first time of the GP should be at the 11$^{th}$ day i.e.<br>$t_1=9000(0.85)^9(1.05)$ |



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Consider

$189000(0.85)^9(1.05^n-1) \ge 90000-48187.53574$

$1.05^n \ge 1.9551545048$

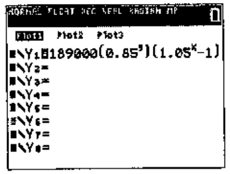
$n \ge 13.74189311$

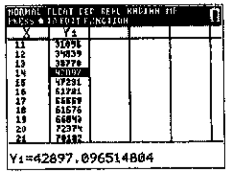
With the change, robot B will take $14+10=24$ days to fill up the tank while robot A still takes 16 days, therefore robot A would be faster in filling up the tank.

**<u>Alternative Method:</u>**

$189000(0.85)^9(1.05^n-1) \ge 90000-48187.53574$

$189000(0.85)^9(1.05^n-1) \ge 41812.46426$






| $n$ | $189000(0.85)^9(1.05^n-1)$ |
|---|---|
| 13 | 38769.83 |
| 14 | $42897.10 > 41812.46426$ |

With the change, robot B will take $14+10=24$ days to fill up the tank while robot A still takes 16 days, therefore robot A would be faster in filling up the tank.

**Comments:** Many students did not recognize that $S_n = 189000(0.85)^9(1.05^n-1)$ was referring to the total volume <u>after</u> <u>reprogramming</u> and thus when finding when the tank will be filled up, there is a need to consider the remaining volume that needs to be filled up i.e. solve $S_n \ge 90000-48187.53574$

Many incorrectly solved the inequality $S_n \ge 90000$ instead.

Another common mistake was incorrectly concluding that B will take 14 days to fill up the tank instead of 14 + 10 = 24 days.





9 The closed curve $C$, which is symmetrical about the line $x=0$, has parametric equations

$$x=\cos 3t+\cos t, \qquad y=-2\cos 2t,$$

for $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

**(a)** Sketch $C$. [1]

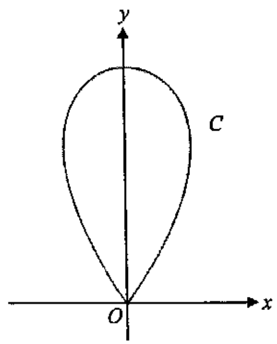
**(b)** Find the exact equation of the tangent of $C$ at the point when $t=\frac{\pi}{4}$. [3]

**(c)** Find the acute angle between the two tangents of curve $C$ at $t=\frac{\pi}{4}$ and $t=\frac{3\pi}{4}$. [2]

**(d)** Show that the area enclosed by the curve $C$ is given by

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 3\sin 5t+\sin 3t+2\sin t \, dt.$$

Hence find the area enclosed by the curve $C$ correct to 3 decimal places. [4]

| 9 | Solution [10] Parametric Curve, Applications of Differentiation, Applications of Integration. | Comments |
|---|---|---|
| (a) |  | Generally ok for this part except for some students who have overlooked the given range of values for $t$ being only $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$ and thus wrongly provided further sketch of non-required portions of the curve. |
| (b) | $\frac{dx}{dt}=-3\sin 3t-\sin t$ and $\frac{dy}{dt}=4\sin 2t$<br>$\frac{dy}{dx}=-\frac{4\sin 2t}{3\sin 3t+\sin t}$<br>When $t=\frac{\pi}{4}$, $x=y=0$ and $\frac{dy}{dx}=-\frac{4\sin\left(\frac{2\pi}{4}\right)}{3\sin\left(\frac{3\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)}=-\sqrt{2}$<br>Equation of the tangent:<br>$y-0=-\sqrt{2}(x-0)$<br>$y=-\sqrt{2}x$. | There were mistakes in differentiating $x$ and $y$ with respect to $t$ resulting in the wrong expression for $\frac{dy}{dx}$ and thus its value when $t=\frac{\pi}{4}$.<br>Also, there were mistake in finding the value of $x$ and $y$ when $t=\frac{\pi}{4}$ and resulted in the wrong equation for the tangent as needed. |

| | | |
|---|---|---|
| | | Students should also simplify their answers eg $\left.\frac{dy}{dx}\right\|_{t=\frac{\pi}{4}}=-\frac{2}{\sqrt{2}}=-\sqrt{2}$ |
| (c) | At $t=\frac{\pi}{4}$, the gradient of the tangent $=-\sqrt{2}$.<br><br>Angle it makes with the $y$-axis $=\frac{\pi}{2}-\tan^{-1}\left(\sqrt{2}\right)$.<br><br>As the curve is symmetric about $x=0$,<br>Required angle<br>$=2\left(\frac{\pi}{2}-\tan^{-1}\sqrt{2}\right)=1.23$ rad(3 s.f.) or 70.5°(1 d.p.) | Not well done for this part. Many students attempted to find the gradient of the tangent at $t=\frac{3\pi}{4}$ to be $\sqrt{2}$ and wrongly conclude that the tangents at $t=\frac{\pi}{4}$ and $t=\frac{3\pi}{4}$ are perpendicular and thus angle between them is 90°.<br>Note that $\sqrt{2}\times\left(-\sqrt{2}\right)\neq-1$. |
| (d) | Required area<br>$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} y(t)\frac{dx}{dt}\,dt$<br>$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} -2\cos 2t\left(-3\sin 3t-\sin t\right)\,dt$<br>$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 3\left(2\sin 3t\cos 2t\right)+2\sin t\cos 2t\,dt$<br>$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 3\left(\sin 5t+\sin t\right)+\sin 3t+\sin\left(-t\right)\,dt$<br>$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 3\sin 5t+\sin 3t+2\sin t\,dt$ (Shown)<br>$=1.508$ units$^2$ (3 d.p.) | Some students were not able to start this part with either the $\int x\left(\frac{dy}{dt}\right)dt$ or $\int y\left(\frac{dx}{dt}\right)dt$ method.<br>For the $\int x\left(\frac{dy}{dt}\right)dt$ method, the required area should be $=-2\int_0^2 x\,dy$<br>$=-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\cos 3t+\cos t\right)\left(4\sin 2t\right)dt$<br>To show the given result, it is more efficient using the $\int y\left(\frac{dx}{dt}\right)dt$ method and apply **factor theorem** to the expressions: $6\sin 3t\cos 2t$ and $2\sin t\cos 2t$.<br>Many students did not realise that for the last part, GC could be use and chose instead to integrate all the involved terms resulting in taking longer time to attain the answer.<br><br>Many students also did not realise that the final answer of this part should be corrected **to 3 decimal places** and left their answer in 3 significant figures instead. |





**10** In a robotics competition, toy cars move along straight lines to complete tasks. Points are defined relative to the origin $(0,0,0)$. The $x$-, $y$- and $z$-axes are in the directions east, north and vertically upwards respectively, with units in centimetres.

The position vectors of two toy cars $A$ and $B$, with respect to time $t$ in seconds, are given as $\mathbf{r}_A=t(5\mathbf{i}+\mathbf{k})$ and $\mathbf{r}_B=\mathbf{i}+6\mathbf{j}-\mathbf{k}+t(4\mathbf{i}-\mathbf{j}+\mathbf{k})$ respectively.

**(a)** Show that after two seconds, car $B$ is at the point with coordinates $(9,4,1)$ and find the distance that car $A$ has travelled in the same duration. [2]

**(b)** Determine whether cars $A$ and $B$ meet. [3]

**(c)** Explain why cars $A$ and $B$ travel on a common plane surface and show that the cartesian equation of the surface is $x-y-5z=0$. [5]

A drone flies above the cars to capture images of the cars during the competition. The shortest distance between the drone and the surface where cars $A$ and $B$ travel is maintained at 50 cm.

**(d)** Find the cartesian equation of the plane containing the flight path of the drone. [2]

| 10 | Solutions [12] 3-D Vectors | Comments |
|---|---|---|
| (a) | At $t=2$, $\mathbf{r}_B=\begin{pmatrix}1\\6\\-1\end{pmatrix}+2\begin{pmatrix}4\\-1\\1\end{pmatrix}=\begin{pmatrix}9\\4\\1\end{pmatrix}$<br>Hence, car $B$ is at point $(9,4,1)$. (shown)<br><br>At $t=2$, $\mathbf{r}_A=2\begin{pmatrix}5\\0\\1\end{pmatrix}=\begin{pmatrix}10\\0\\2\end{pmatrix}$<br>Distance that car $A$ has travelled<br>$=\left\|\begin{pmatrix}10\\0\\2\end{pmatrix}\right\|$<br>$=\sqrt{10^2+2^2}$<br>$=\sqrt{104}$ cm | Vast majority are able to show that car $B$ is a the position. A few students **verify** that $t=2$ instead which is not acceptable.<br><br>For the second part, while most students are able to obtain $\mathbf{r}_A=\begin{pmatrix}10\\0\\2\end{pmatrix}$, some of them interpreted the subsequent distance wrongly, with many of them finding $\left\|\begin{pmatrix}10\\0\\2\end{pmatrix}\right\|-\left\|\begin{pmatrix}5\\0\\1\end{pmatrix}\right\|$ instead. |
| (b) | When the 2 cars meet, they have travelled the same amount of time $t$,<br>$t\begin{pmatrix}5\\0\\1\end{pmatrix}=\begin{pmatrix}1\\6\\-1\end{pmatrix}+t\begin{pmatrix}4\\-1\\1\end{pmatrix}$<br>$\Rightarrow\begin{cases}5t=1+4t---(1)\\0=6-t---(2)\\t=-1+t---(3)\end{cases}$<br>From (2), $t=6$<br>However, it does not satisfy (1) and (3).<br>No consistent solution. | Many who solve the question this way managed to conclude correctly. However, there are some students who concluded that the cars do not meet because the paths do not intersect. That is not correct. **Note that the paths of the cars actually cross each other; it's just that the cars passed by the point of intersection at different times and hence do not meet.** |



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| | Solution | Comments |
|---|---|---|
| | Hence cars $A$ and $B$ do not meet.<br><br>**Alternatively,**<br>$t_A\begin{pmatrix}5\\0\\1\end{pmatrix}=\begin{pmatrix}1\\6\\-1\end{pmatrix}+t_B\begin{pmatrix}4\\-1\\1\end{pmatrix}\Rightarrow\begin{cases}5t_A=1+4t_B\\0=6-t_B\\t_A=-1+t_B\end{cases}$<br>Solving, $t_A=5$ and $t_B=6$<br>Since $t_A\neq t_B$, cars $A$ and $B$ do not meet.<br>(Although the two paths intersect.) | For students who solved the question this way (having different symbols for the parameters), more concluded wrongly because they thought that because the paths intersect, the cars meet. |
| (c) | To check if the 2 lines of travel lie on the same plane,<br>$t_A\begin{pmatrix}5\\0\\1\end{pmatrix}=\begin{pmatrix}1\\6\\-1\end{pmatrix}+t_B\begin{pmatrix}4\\-1\\1\end{pmatrix}\Rightarrow\begin{cases}5t_A=1+4t_B\\0=6-t_B\\t_A=-1+t_B\end{cases}$<br>Solving, $t_A=5$ and $t_B=6$<br>Therefore, the lines intersect.<br><br>Since the lines intersect (i.e. the paths of cars $A$ and $B$ intersect), hence the two cars travel on a common surface.<br><br>Normal to surface $=\begin{pmatrix}5\\0\\1\end{pmatrix}\times\begin{pmatrix}4\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\-5\end{pmatrix}$<br>$\therefore$ the cartesian equation of the common surface is $x-y-5z=0$.<br><br>**Alternative**<br>$\begin{pmatrix}5\\0\\1\end{pmatrix}\times\begin{pmatrix}4\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\-5\end{pmatrix}$<br>Consider the plane $p$: $x-y-5z=0$.<br>As,<br>$\begin{pmatrix}5\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-5\end{pmatrix}=0$ and $\begin{pmatrix}4\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-5\end{pmatrix}=0$<br>Both line of travel is parallel to the plane.<br><br>As, (0,0,0) is on the line of travel of car A and<br>$\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-5\end{pmatrix}=0$, (0,0,0) is on plane $p$ as well.<br>Therefore, the line of travel of car A lies on plane $p$.<br><br>As, (1, 6, −1) is on the line of travel of car B and | Many students could not earn the full credits for this question because they could not argue correctly why the paths travel on a common plane. Many, however, were able to find the normal vector of the common plane and set up its equation. |





| | Solution | Comments |
|---|---|---|
| | $\begin{pmatrix}1\\6\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-5\end{pmatrix}=1-6+5=0$, (1, 6, −1) is on plane $p$ as well.<br>Therefore, the line of travel of car B lies on plane $p$.<br><br>Therefore, both path of travel is on a common plane $x-y-5z=0$. | |
| (d) | Let the equation of the required plane be<br>$\mathbf{r}\cdot\begin{pmatrix}1\\-1\\-5\end{pmatrix}=d$, i.e. $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\-5\end{pmatrix}=\frac{d}{\sqrt{27}}$<br>$\therefore\left\lvert\frac{d}{\sqrt{27}}\right\rvert=50$<br>$\Rightarrow\frac{d}{\sqrt{27}}=-50$<br>(not 50 because the normal vector's $z$-coordinate is negative, indicating it is pointing downwards)<br>$\therefore d=-150\sqrt{3}$<br><br>Hence, the equation of the plane is<br>$x-y-5z=-150\sqrt{3}$ | This part is not well answered.<br><br>Amongst students who can handle distances between parallel planes, almost all of them fail to realise the need for the negative sign for '$-150\sqrt{3}$'. Only one or two students realised that! |

**11** An experiment was conducted at room temperature, where the levels of the concentration of a chemical is investigated over time.
The initial concentration of the chemical was $x_0$ mol/dm$^3$. A possible model suggests that the rate at which the concentration decreases is directly proportional to $x^2$, where $x$ mol/dm$^3$ is the concentration of the chemical at time $t$ minutes after the start of the experiment.

**(a)** **(i)** By setting up and solving a differential equation, show that the time taken for the concentration of the chemical to reach $\frac{x_0}{2}$ is inversely proportional to $x_0$. [4]

**(ii)** It was observed that it took 4 min and 16 min to reach one-half and one-quarter of $x_0$ respectively. Explain why the above model is not suitable. [2]

For the rest of the question, take $x_0 = \frac{3}{2}$.

It was later discovered that the concentration of the chemical can be modelled in an alternative way. Due to a reversible reaction, the rate at which the concentration of the chemical increases is directly proportional to $\left(\frac{3}{2}-x\right)^2$ while the rate at which it decreases is directly proportional to $x^2$.

It is given that there is no change in the concentration when the concentration is $\frac{1}{2}$ mol/dm$^3$.

**(b)** **(i)** For this model, show that $\frac{dx}{dt} = -k\left(4x^2+4x-3\right)$, where $k$ is a positive real constant. [3]

**(ii)** Solve this differential equation to find $x$ in terms of $t$ and $k$. [5]

| 11 | Solution [14] Differential Equations | Comments |
|---|---|---|
| (ai) | $\frac{dx_{decrease}}{dt} \propto x^2$<br>$\frac{dx_{decrease}}{dt} = ax^2$, for $a>0$<br>$\frac{dx}{dt} = -ax^2$<br>$\frac{1}{x^2}\frac{dx}{dt} = -a$<br>$\int \frac{1}{x^2}\,dx = \int -a\,dt$<br>$\frac{-1}{x} = -at + C$<br>$x = \frac{-1}{C-at}$<br>When $t=0, x=x_0$.<br>$x_0 = \frac{-1}{C} \Rightarrow C = -\frac{1}{x_0}$ | Mixed responses.<br><br>A good number of students were able to identify and write down the DE.<br><br>However, a good minority were not able solve the integral $\int \frac{1}{x^2}\,dx$, leaving **incorrect** results such as $\ln\lvert x^2\rvert$ and $\frac{x^{-3}}{-3}$.<br><br>And another good minority did not sub in the initial condition of $t=0, x=x_0$ to solve for the arbitrary constant $C$. |





| | | |
|---|---|---|
| | $x = \frac{-1}{-\frac{1}{x_0} - at}$<br>$x = \frac{x_0}{1+ax_0t}$<br>When $x = \frac{x_0}{2}$,<br>$\frac{x_0}{2} = \frac{x_0}{1+ax_0t}$<br>$1+ax_0t = 2$<br>$ax_0t = 1$<br>$t = \left(\frac{1}{a}\right)\left(\frac{1}{x_0}\right)$ $\left(\text{with } \frac{1}{a} > 0\right)$<br>$t \propto \frac{1}{x_0}$ (Shown) | Stronger responses were able to identify and write the required DE and solve for the general solution. Next, they would have subbed in the initial condition given then sub in $x = \frac{x_0}{2}$ in their equation to show that $t = k\left(\frac{1}{x_0}\right)$ for some real constant $k$ before concluding that $t \propto \frac{1}{x_0}$. |
| (aii) | Time taken for decay of $x_0$ to $\frac{1}{2}x_0 = 4$ min.<br>Time taken for decay of $\frac{1}{2}x_0$ to $\frac{1}{4}x_0 = 16-4 = 12$ min.<br>From (ai), if the model was followed, as the time taken is inversely proportional, we would have:<br>Time taken for decay of $\frac{1}{2}x_0$ to $\frac{1}{4}x_0 = 2(4) = 8$ min instead.<br>Hence it does not follow the model in (ai).<br><br>**<u>Alternative Method:</u>**<br>When $x = \frac{x_0}{2}, t = 4$<br>$\frac{x_0}{2} = \frac{x_0}{1+4ax_0}$<br>$a = \frac{1}{4x_0}$<br>But when $x = \frac{x_0}{4}, t = 16$<br>$\frac{x_0}{4} = \frac{x_0}{1+16ax_0}$<br>$a = \frac{3}{16x_0} \neq \frac{1}{4x_0}$<br>This is not possible as $a$ is a fixed constant. | The question proved challenging to most students.<br><br>Many were not sure how to use the timings given to show that the model was not appropriate.<br><br>Many who tried to used **part (ai)**'s result did not fully appreciate what the result meant. Many wrote workings similar to comparing 8 mins with 16 mins, which was not correct as the second half-life was $16-4=12$ mins.<br><br>Attempts that subbed the timings back into their equation were much longer and saw more success. |

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| (b) | $\frac{dx_{\text{increase}}}{dt} \propto \left(\frac{3}{2}-x\right)^2$<br>$\frac{dx_{\text{increase}}}{dt} = b\left(\frac{3}{2}-x\right)^2$, for $b>0$<br>$\frac{dx}{dt} = \frac{dx_{\text{increase}}}{dt} - \frac{dx_{\text{decrease}}}{dt}$<br>$\frac{dx}{dt} = b\left(\frac{3}{2}-x\right)^2 - ax^2$<br>When $\frac{dx}{dt}=0, x=\frac{1}{2}$,<br>$0 = b\left(\frac{3}{2}-\frac{1}{2}\right)^2 - a\left(\frac{1}{2}\right)^2$<br>$0 = b - \frac{1}{4}a$<br>$a = 4b$<br>$\frac{dx}{dt} = b\left(\frac{3}{2}-x\right)^2 - 4bx^2$<br>$\frac{dx}{dt} = b\left(\frac{9}{4}-3x+x^2-4x^2\right)$<br>$\frac{dx}{dt} = k\left(3-4x-4x^2\right)$, $k=\frac{3}{4}b$<br>$\frac{dx}{dt} = -k\left(4x^2+4x-3\right)$ (Shown) | Generally okay, but could be better.<br><br>A strong minority assumed that the proportionality constant was the same throughout and wrote $\frac{dx}{dt} = k\left[\left(\frac{3}{2}-x\right)^2 - x^2\right]$ which was **incorrect**.<br><br>The best of responses where critical that ***k* needed to be positive** and **properly defined their proportionality constants at the beginning**. |
| (c) | $4x^2+4x-3 = (2x+1)^2-4$<br>$\frac{dx}{dt} = -k\left(4x^2+4x-3\right)$<br>$\frac{dx}{dt} = -k\left((2x+1)^2-4\right)$<br>$\frac{1}{(2x+1)^2-2^2}\frac{dx}{dt} = -k$<br>$\frac{1}{2}\int \frac{2}{(2x+1)^2-2^2}\,dx = \int -k\,dt$<br>$\frac{1}{2}\left(\frac{1}{2(2)}\ln\left\lvert\frac{2x+1-2}{2x+1+2}\right\rvert\right) = -kt + C_0$<br>$\ln\left\lvert\frac{2x-1}{2x+3}\right\rvert = -8kt + 8C_0$<br>$\frac{2x-1}{2x+3} = \pm e^{-8kt+8C_0} = De^{-8kt}$, $D = \pm e^{8C_0}$<br>When $t=0, x_0=\frac{3}{2}$ | Mixed responses.<br><br>Majority of the students had the appropriate strategy (either use partial fractions or completing the square) to tackle the problem. However, many did not have the algebraic skills/routines to solve the question to completion.<br><br>The strongest of responses showed algebraic flare and followed the routine with finesse. Most importantly they remembered the $\lvert . \rvert$ brackets after integration and made $x$ the subject accurately.<br><br>Common Mistakes include: |





| | | |
|---|---|---|
| | $\frac{2\left(\frac{3}{2}\right)-1}{2\left(\frac{3}{2}\right)+3} = De^0$<br>$D = \frac{4}{12} = \frac{1}{3}$<br>$\frac{2x-1}{2x+3} = \frac{1}{3}e^{-8kt}$<br>$6x-3 = 2e^{-8kt}x + 3e^{-8kt}$<br>$x\left(6-2e^{-8kt}\right) = 3e^{-8kt}+3$<br>$x = \frac{3+3e^{-8kt}}{6-2e^{-8kt}}$<br><br>**Alternative Method:**<br>$\frac{dx}{dt} = -k\left(4x^2+4x-3\right)$<br>$\frac{dx}{dt} = -k(2x-1)(2x+3)$<br>$\frac{1}{(2x-1)(2x+3)}\frac{dx}{dt} = -k$<br>$\frac{1}{4}\int \frac{1}{2x-1} - \frac{1}{2x+3}\,dx = \int -k\,dt$<br>$\frac{1}{4}\left(\frac{1}{2}\ln\lvert 2x-1\rvert - \frac{1}{2}\ln\lvert 2x+3\rvert\right) = -kt + C_0$<br>$\ln\left\lvert\frac{2x-1}{2x+3}\right\rvert = -8kt + 8C_0$<br>$\frac{2x-1}{2x+3} = \pm e^{-8kt+8C_0} = De^{-8kt}$, $D = \pm e^{8C_0}$<br>When $t=0, x_0=\frac{3}{2}$<br>$\frac{2\left(\frac{3}{2}\right)-1}{2\left(\frac{3}{2}\right)+3} = De^0$<br>$D = \frac{4}{12} = \frac{1}{3}$<br>$\frac{2x-1}{2x+3} = \frac{1}{3}e^{-8kt}$<br>$6x-3 = 2e^{-8kt}x + 3e^{-8kt}$<br>$x\left(6-2e^{-8kt}\right) = 3e^{-8kt}+3$<br>$x = \frac{3+3e^{-8kt}}{6-2e^{-8kt}}$ | - Not completing the square properly.<br>- Not factorising properly: $4x^2+4x-3 \neq (x-0.5)(x+3.5)$.<br>- Ignoring the coefficient of $x$ and $\lvert . \rvert$ when integrating:<br>$\int \frac{1}{2x-1}\,dx \neq \ln(2x-1)+c$<br>$\int \frac{1}{2x-1}\,dx = \frac{1}{2}\ln\lvert 2x-1\rvert + c$<br>**Or**<br>$\int \frac{1}{(2x+1)^2-2^2}\,dx \neq \frac{1}{2(2)}\ln\left(\frac{2x+1-2}{2x+1+2}\right)+c$<br>$\int \frac{1}{(2x+1)^2-2^2}\,dx = \frac{1}{2}\left(\frac{1}{2(2)}\ln\left\lvert\frac{2x+1-2}{2x+1+2}\right\rvert\right)+c$<br>- Not using the initial condition $t=0, x=\frac{3}{2}$.<br>- Not making $x$ the subject, even though the question request to **find *x*** in terms of $k$ and $t$. |

**END OF PAPER**

# Solution and Comments for 2024 H2 Math Prelim P2

**Section A: Pure Mathematics [40 marks]**

1 **(a)** Find $\int \ln x \, dx$. [2]

**(b)** The region $A$ is bounded by the curve $y=\frac{1}{2}\sqrt{\ln x}$, $x$-axis and the line $x=5$. Find the exact volume when $A$ is rotated $2\pi$ radians about the $x$-axis. [3]

| 1 | Solutions [5] Integration | Comments |
|---|---|---|
| (a) | $\int \ln x \, dx$<br>$=\int (1)\ln x \, dx$<br>$= x\ln x - \int \left(\frac{1}{x}\right) x \, dx$    $u=\ln x$, $\frac{du}{dx}=\frac{1}{x}$, $\frac{dv}{dx}=1$, $v=x$<br>$= x\ln x - \int 1 \, dx$<br>$= x\ln x - x + c$ | There is mixed responses from the students. There's a group of students who $\int \ln x \, dx = \frac{1}{x}+C$ which is clearly incorrect. There is another group of students who left out the constant of integration and they were not awarded the full marks. |
| (b) | Volume<br>$=\pi\int_1^5 \left(\frac{1}{2}\sqrt{\ln x}\right)^2 dx$<br>$=\frac{\pi}{4}\int_1^5 \ln x \, dx$<br>$=\frac{\pi}{4}\left[x\ln x - x\right]_1^5$<br>$=\frac{\pi}{4}(5\ln 5 - 5 + 1)$<br>$=\frac{\pi}{4}(5\ln 5 - 4)$ units$^3$ | Students should write clearly their workings and no marks will be awarded for content which the markers are not able to read. Some students did not simplify their final answer and have marks deducted as well.<br><br>Students should make use of their calculator for evaluation of their answer (if required) and should not penalise themselves, for example $5^5 \neq 25$! |





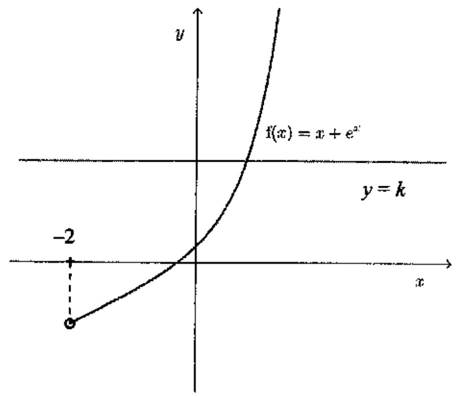
2 Functions f and g are defined by

$$\mathrm{f}: x \mapsto x + e^x, \text{ for } x \in \mathbb{R}, x > -2,$$
$$\mathrm{g}: x \mapsto \ln x \text{ for } x \in \mathbb{R}, x > \frac{1}{e}.$$

**(a)** Show that f has an inverse. [1]

**(b)** Show that the composite function fg exist and find $\mathrm{fg}(x)$. [3]

**(c)** Hence find the value of $x$ which satisfies $\mathrm{g}(x)=\mathrm{f}^{-1}(x)$. [3]

| 2 | Solution [7] Functions | Comments |
|---|---|---|
| (a) | <br>Any horizontal line $y=k$, where $k\in\mathbb{R}$ cuts the graph of $y=\mathrm{f}(x)$ at most once. Therefore, f is a one to one function. Thus, f has an inverse. | Generally well-done for students who used "at most once". Those who used "exactly" once tended to make mistake when a line they proposed (e.g $y = k$, where $k > -2$) that does not even cut the graph of $y$ = f($x$) when $k$ = -1.9. |
| (b) | $R_g=(-1,\infty)\subseteq(-2,\infty)=D_f$<br>Therefore, the composite function fg exist.<br>$\mathrm{fg}(x)=\mathrm{f}(\ln x)$<br>$=\ln x + e^{\ln x}$<br>$=\ln x + x$. | Generally well-done. |
| (c) | $\mathrm{g}(x)=\mathrm{f}^{-1}(x)$<br>$\mathrm{fg}(x)=x$<br>$\ln x + x = x$<br>$\ln x = 0$<br>$x=1$ | Many did not realise that both sides can be composed with f so that part (b) expression for fg can be used.<br>Some attempted to find f$^{-1}$ but was not successful. |
| | | |

3 A function f is defined by $f(x)=ax+b+\frac{c}{x-1}$, where $a$, $b$ and $c$ are constants. The graph of $y=f(x)$ has a minimum point at $(2.5,13)$ and also passes through the $y$-axis at $(0,-12)$.

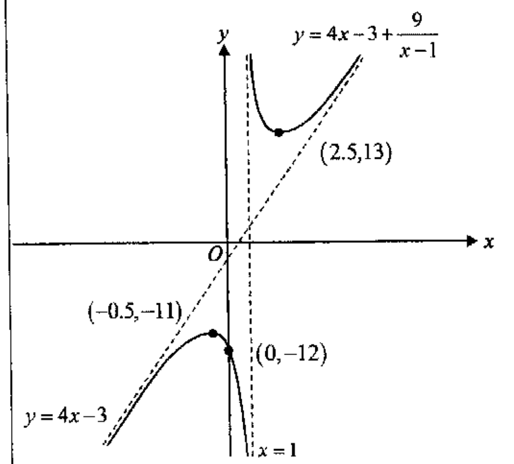
**(a)** Find the values of $a$, $b$ and $c$. [4]

**(b)** Sketch the graph of $y=f(x)$, stating clearly any asymptotes, axial intercepts and turning points. [3]

| 3 | Solutions [7] LSoE + Graphing Techniques | Comments |
|---|---|---|
| (a) | $f(0)=-12$<br>$b+\frac{c}{-1}=-12$<br>$b-c=-12$<br><br>$f(2.5)=13$<br>$2.5a+b+\frac{c}{2.5-1}=13$<br>$2.5a+b+\frac{2}{3}c=13$<br><br>$f(x)=ax+b+\frac{c}{x-1}$<br>$f'(x)=a-\frac{c}{(x-1)^2}$<br>$f'(2.5)=0$<br>$a-\frac{c}{(2.5-1)^2}=0$<br>$a-\frac{4}{9}c=0$<br>Solving,<br>$a=4$, $b=-3$ and $c=9$. | This question ought to be standard and manageable but quite a significant number of students fail to obtain the correct values for $a$, $b$ and $c$.<br>They either made errors in substitution or calculations, or they differentiated f($x$) wrongly. |





| (b) | | |
|---|---|---|
| |  | Most students were able to get the correct shape of the graph and the asymptotes (even if values of $a$, $b$ and $c$ were found wrongly in the part (a)).<br><br>Some students identified the position of the maximum point $x=1$ wrongly, thinking that it is to the right of the $y$-intercept (-12). A few others thought that (0, -12) is the maximum point.<br><br>A few students did not reflect well that asymptotic behaviour of the graph. |

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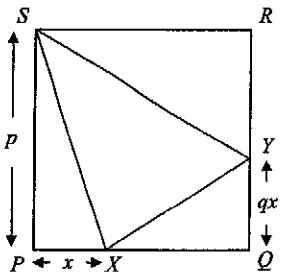



The diagram shows a square $PQRS$ of side $p$ metres. The points $X$ and $Y$ lie on $PQ$ and $QR$ respectively such that $PX = x$ m and $QY = qx$ m, where $q$ is a constant such that $q > 1$.

**(a)** Given that the area of triangle $XYS$ is $A$ m$^2$, show that $A = \frac{1}{2}\left(qx^2 - px + p^2\right)$. [3]

**(b)** Given that $x$ can vary, show that $QY = YR$ when $A$ is minimum and express the minimum value of $A$ in terms of $p$ and $q$. [6]

| 4 | Solution [9] Applications of Differentiation | Comments |
|---|---|---|
| (a) | $A =$ Area of $PQRS -$ area of $\Delta SPX - \Delta QXY - \Delta RSY$<br>$= p^2 - \frac{1}{2}xp - \frac{1}{2}xq(p-x) - \frac{1}{2}p(p-qx)$<br>$= p^2 + \frac{1}{2}\left(-xp - pqx + qx^2 - p^2 + pqx\right)$<br>$= \frac{1}{2}\left(qx^2 - px + p^2\right)$.<br>(shown) | Fairly well attempted.<br>There were some with quite difficult to read handwriting and did not show the steps clearly.<br><br>Some went to find the height and base of triangle *XYS* and its not a feasible method to prove the required result. |
| (b) | $p$ and $q$ are constants, we differentiate $A$ wrt $x$,<br><br>$\frac{dA}{dx} = \frac{1}{2}(2qx - p)$<br>Let $\frac{dA}{dx} = 0$<br>$x = \frac{p}{2q}$<br>Hence $PX = \frac{p}{2q}$ & $QY = q(\frac{p}{2q}) = \frac{p}{2}$<br>Since $RQ$ and $SP$ are sides of a square, $RQ = SP = p$<br>$YR = RQ - QY = p - QY = \frac{p}{2} = QY$ (shown)<br>$\frac{d^2A}{dx^2} = \frac{1}{2}(2q) = q > 1 > 0$ (so $A$ is minimum) | This is quite a straight forward question.<br><br>There was quite a number who just proclaimed that *YR*=*QY* after obtaining $2qx = p$, which is not allowed as this is a shown question; so more details are needed to show explicitly.<br>Students are reminded that the question requires you to prove $QY = YR$, and this is not a fact to use it at the start of the question.<br>Many students forgot to use 2[nd] derivative to prove that |





| | | |
|---|---|---|
| | now, minimum $A$ occurs when $x = \frac{p}{2q}$<br>$A = \frac{1}{2}\left(q\left(\frac{p}{2q}\right)^2 - p\left(\frac{p}{2q}\right) + p^2\right)$<br>$= \frac{1}{2}\left(\frac{p^2}{4q} - \frac{p^2}{2q} + p^2\right)$<br>$= \frac{p^2}{2}\left(1 - \frac{1}{4q}\right)$ | A is minimum. And also not ascertaining why $q > 0$ (the reason is because it is given $q > 1$, so it has to be $> 0$).<br>Note : those who just declare 2[nd] derivative $> 0$ without a reason will have 1 mark deducted.<br><br>Most students know to substitute and put into the area $A$ expression in (a). However the algebraic manipulation is very badly done with all sorts of mistakes. A significant number were able to reach $\frac{1}{2}\left(\frac{p^2}{4q} - \frac{p^2}{2q} + p^2\right)$ but went on to simplify wrongly, which means they are not awarded the full credit!<br><br>As a general rule of thumb, if the final answer after simplifying is wrong, full marks will not be earned even though the intermediate working was correct. |



BP-609

5 **Do not use a calculator in answering this question.**

Let $\mathrm{P}(z)$ be a polynomial with real coefficients and $z = re^{i\theta}$ is one the roots of the equation $\mathrm{P}(z) = 0$.

**(a)** Show that $z^2 - (2r\cos\theta)z + r^2$ is a quadratic factor of $\mathrm{P}(z)$. [3]

**(b)** Given that $z^4 - z^3 + z^2 - z + 1 = (z^2 - az + 1)(z^2 - bz + 1)$, for real values $a$ and $b$, and that $b < 0 < a$, find exact values for $a$ and $b$. [4]

**(c)** By considering $1 + z^5$, verify that $z = e^{i\frac{\pi}{5}}$ is a root to the equation $z^4 - z^3 + z^2 - z + 1 = 0$. [2]

**(d)** Show that $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$. [3]

| 5 | Solution [12] Complex Number + APGP | Comments |
| --- | --- | --- |
| **(a)** | Since $z = re^{i\theta}$ is a root to the equation $\mathrm{P}(z) = 0$, $(z - re^{i\theta})$ is a factor of $\mathrm{P}(z)$<br>As all coefficients of $\mathrm{P}(z) = 0$ are real and $z = re^{i\theta}$ is a root, $z^* = re^{i(-\theta)}$ is also a root.<br>Thus $(z - re^{i(-\theta)})$ is also a factor of $\mathrm{P}(z)$.<br>Multiplying both factors together, we get a quadratic factor of $\mathrm{P}(z)$:<br>$(z - re^{i\theta})(z - re^{i(-\theta)}) = z^2 - re^{i\theta}z - re^{i(-\theta)}z + (re^{i\theta})(re^{i(-\theta)})$<br>$= z^2 - r(e^{i\theta} + e^{i(-\theta)})z + r^2e^{i\theta + i(-\theta)}$<br>$= z^2 - r(2\mathrm{Re}(e^{i\theta}))z + r^2e^{i(\theta-\theta)}$<br>$= z^2 - 2rz\cos\theta + r^2$ (shown) | A few students did not explain appropriately why the conjugate of $z$ is also a root of the equation.<br><br>Instead of forming the quadratic factor by the product of the linear factors, some students formed with the product of the roots $(z \times z^*)$ instead! |
| **(b)** | $z^4 - z^3 + z^2 - z + 1 = (z^2 - az + 1)(z^2 - bz + 1)$<br>Comparing coefficient of $z$ or $z^3$,<br>$-a - b = -1$<br>$a + b = 1$<br>Comparing coefficient of $z^2$,<br>$2 + ab = 1$<br>$ab = -1$<br>Solving,<br>$a(1-a) = -1$<br>$a^2 - a - 1 = 0$<br>$a = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$<br>As $b < 0 < a$, $a = \frac{1+\sqrt{5}}{2}$, $b = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$.<br>Therefore,<br>$z^4 - z^3 + z^2 - z + 1 = \left(z^2 - \frac{1+\sqrt{5}}{2}z + 1\right)\left(z^2 - \frac{1-\sqrt{5}}{2}z + 1\right)$ | This part is generally well done, with a few students making errors in calculations or not realising that we cannot use calculators to find the roots here. |
| **(c)** | If $z = e^{i\left(\frac{\pi}{5}\right)}$,<br>$1 + z^5 = 1 + e^{i\left(\frac{\pi}{5}\right)5}$<br>$= 1 + e^{i(\pi)}$<br>$= 1 - 1$<br>$= 0$.<br><br>$z^4 - z^3 + z^2 - z + 1$ is a GP where<br>first term = 1, common ratio = $-z$, number of terms = 5<br>Therefore, when $z = e^{i\left(\frac{\pi}{5}\right)}$,<br>$z^4 - z^3 + z^2 - z + 1 = \frac{1(1 - (-z)^5)}{1 - (-z)}$<br>$= \frac{1 + z^5}{1 + z}$<br>$= \frac{0}{1 + z}$<br>$= 0$. (Verified) | This question was not well answered.<br>Many students multiply the given equation with $z$ to $z^4 - z^3 + z^2 - z + 1 = 0$ to get $z^5 - z^4 + z^3 - z^2 + z = 0$ but this is inappropriate: multiplying factor(s) to an equation would possibly introduce new root(s) to the equation and hence produces wrong analysis. |
| **(d)** | Let $\mathrm{P}(z) = z^4 - z^3 + z^2 - z + 1$.<br>From **(c)** we have $z = e^{i\left(\frac{\pi}{5}\right)}$.<br>Thus, from part **(a)**, a quadratic factor of $\mathrm{P}(z) = 0$ is $z^2 - 2(1)z\cos\frac{\pi}{5} + (1)^2 = z^2 - 2\cos\frac{\pi}{5}z + 1$.<br>Since both factors in part **(b)** have only real coefficients, and $z = e^{i\left(\frac{\pi}{5}\right)}$ is indeed a root of one of them, one quadratic factor.<br>Hence, comparing to the result from part **(b)**,<br>Either $2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$ or $2\cos\frac{\pi}{5} = \frac{1-\sqrt{5}}{2}$<br>Since $\frac{\pi}{5}$ is an acute angle, $\cos\frac{\pi}{5} > 0$<br>Therefore,<br>$2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$<br>$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$. (Shown) | Many students did not know how to solve this question or have not considered the 'hence' approach.<br>For those who did, there is a need to explain why $\frac{1+\sqrt{5}}{2}$ was used for comparison rather than $\frac{1-\sqrt{5}}{2}$. |

## Section B: Statistics [60 marks]

6 The number 396900 can be expressed as $2^2 \times 3^4 \times 5^2 \times 7^2$.
A factor of 396900 can be expressed in the form $2^a \times 3^b \times 5^c \times 7^d$ for non-negative integers $a$, $b$, $c$ and $d$. For example, 3150 and 140 are factors of 396900 and they can be expressed as $2^1 \times 3^2 \times 5^2 \times 7^1$ and $2^2 \times 3^0 \times 5 \times 7$ respectively.

**(a)** State all the possible values for $a$. [1]

**(b)** Find the number of factors of 396900. [1]

A factor of 396900 is chosen randomly.

**(c)** Find the probability that the chosen factor is divisible by 3 given that it is even. [3]

| 6 | Solution [5] Probability | Comments |
|---|---|---|
| (a) | 0, 1, 2 | Most students either know how to approach these two parts or are totally clueless. |
| (b) | $a$ has 3 possibilities (namely 0 to 2)<br>$b$ has 5 possibilities (namely 0 to 4)<br>$c$ has 3 possibilities (namely 0 to 2)<br>$d$ has 3 possibilities (namely 0 to 2)<br><br>Total number of factors $= 3\times5\times3\times3 = 135$ | |
| (c) | Number of even factors $= 2\times5\times3\times3 = 90$<br>(As $a$ has only 2 possibilities, 1 or 2)<br><br>Number of even factors that is divisible by 3 $= 2\times4\times3\times3 = 72$.<br>(Now $b$ only has 4 possibilities, namely 1 to 4)<br><br>$\mathrm{P}(\text{factor is divisible by } 3 \mid \text{factor is even})$<br>$= \dfrac{\mathrm{P}(\text{factor is divisible by } 3 \cap \text{factor is even})}{\mathrm{P}(\text{factor is even})}$<br>$= \dfrac{\frac{\mathrm{n}(\text{even factors divisible by } 3)}{\mathrm{n}(\text{all factors})}}{\frac{\mathrm{n}(\text{even factors})}{\mathrm{n}(\text{all factors})}}$<br>$= \dfrac{\mathrm{n}(\text{even factors divisible by } 3)}{\mathrm{n}(\text{even factors})}$<br>$= \dfrac{72}{90} = \dfrac{4}{5}$ | While many students realised the need for conditional probability, quite many of them students could not solve the question completely. |





7 Company A consists of 15 men and 10 women on its staff. 10 staff are to be selected to join a contest as Team A. The random variable $R$ denotes the number of men in Team A.

**(a)** Show that

$$\mathrm{E}(R) = \sum_{r=0}^{10}\left[ r \times \frac{\binom{15}{r}\binom{m}{10-r}}{n} \right], \text{ for } r = 0, 1, 2, \ldots, 10,$$

where $m$ and $n$ are real constants to be determined. [3]

**(b)** Use your calculator to find E($R$) and $\mathrm{Var}(R)$. [3]

Company B also consists of male and female staff where 10 staff are to be selected to join the same contest as Team B, and $Q$ denotes the number of men in the team.
It is known that E($Q$) = 6.25 and Var($Q$) = 3.

Before both companies finalise their teams, they decide to study the different team configurations further by generating a list of random samples of the teams.

**(c)** Estimate the probability the mean number of men in 30 random samples of Team A exceeds the mean number of men in 30 random samples of Team B. [4]

| 7 | Solutions [10] PnC + DRV + Sampling | Comments |
|---|---|---|
| (a) | For $r = 0, 1, 2, \ldots, 10$,<br>The number of ways to choose $r$ men and $10-r$ women<br>$= \binom{15}{r}\binom{10}{10-r}$.<br>Total number of ways to make a team $= \binom{25}{10} = 3268760$<br>So,<br>$\mathrm{P}(R = r)$<br>$= \dfrac{\binom{15}{r}\binom{10}{10-r}}{3268760}$<br><br>$\mathrm{E}(R) = \sum_{r=0}^{10}[r \times \mathrm{P}(R=r)]$<br>$= \sum_{r=0}^{10}\left[ r \times \dfrac{\binom{15}{r}\binom{10}{10-r}}{3268760} \right]$ (Shown) | This question proved challenging to students.<br><br>There was a strong minority that did not attempt the question.<br><br>A few students thought that this was a Binomial Distribution, but the probability of picking a man is not constant, failing the criteria of a Binomial Distribution.<br><br>Many were randomly trying out methods, and many managed to deduce $m$ by sheer guessing.<br><br>Stronger responses recalled that<br>$\mathrm{E}(R) = \sum_{r=0}^{10}[r \times \mathrm{P}(R=r)]$<br>and noticed that the fraction was merely just $\mathrm{P}(R=r)$. |



BP-611

Different strategies were seen here, either listing (not recommended) or showing the general case directly(recommended).

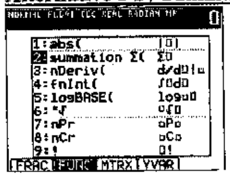
**(b)** From (a)

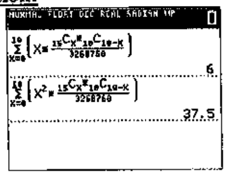
$$E(R)=\sum_{r=0}^{10}\left[r\times\frac{\binom{15}{r}\binom{10}{10-r}}{3268760}\right]=6 \text{ by GC.}$$

$$\text{Var}(R)=E(R^2)-E(R)^2$$
$$=\sum_{r=0}^{10}\left[r^2\times\frac{\binom{15}{r}\binom{10}{10-r}}{3268760}\right]-6^2$$
$$=37.5-36$$
$$=\frac{3}{2}.$$

GC Steps:

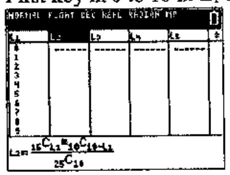
**Alternative 1 by Summation:**

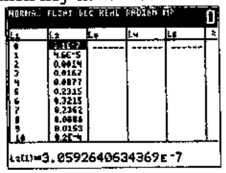



**Alternative 2 by List:**

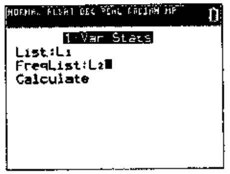
First key in 0 to 10 in $L_1$ then key in the formula in $L_2$.

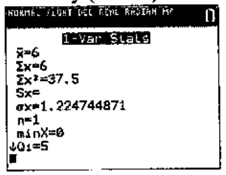





Next got to 1-Var Stats, list as $L_1$ and Freqlist as $L_2$.

Once can deduce the mean directly (i.e. $\bar{x}$ .)




This question proved challenging to students.

There are few that thought that this was Binomial Distribution and applied the formula for that here. Again, it does not apply in this situation.

The summation notation in **part (a)** was a hint to use the GC's summation function. Those who did so, quickly and accurately solved the question.

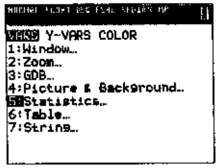
Students are recommended to write the formulas of what needs to be found, as specially for a 'high' mark question.

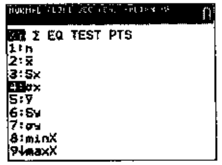




To get the variance, go to 'VARS' and selection option 5:Statistics… .

Then, select $\sigma_X$, this gives you the standard deviation.




Square the value to achieve the variance.


**(c)** Since $n=30$ is large, by CLT,

$\bar{R}\sim N(6,\frac{1}{20})$ approx , $\bar{Q}\sim N(6.25,\frac{1}{10})$ approx

$\bar{R}-\bar{Q}\sim N(-0.25,\frac{1}{20}+\frac{1}{10})=N(-0.25,\frac{3}{20})$ approx

$$P(\bar{R}>\bar{Q})$$
$$=P(\bar{R}-\bar{Q}>0)$$
$$=0.259 \text{ (3 s.f.)}$$

This question proved extremely challenging to many students.

The question hinted at an approximation via the word "Estimate". This should lead one to realise CLT was involved.

Many did not realise that both $R$ and $Q$ are **NOT** normal distributions. Moreover, this was a misconception that CLT implied that both $R$ and $Q$ will become normal distributions **approximately**. Instead, CLT implies that their means, i.e. $\bar{R}$ and $\bar{Q}$ are normal distributions **approximately**.

**8** Long standing data indicates that customers of 80% of all table reservations at a restaurant will turn up. For this question, assume that all tables can only be reserved once in a dinner service and customers stay until the end of dinner service.

**(a)** One day, a restaurant has 14 table reservations for dinner service.

**(i)** Find the expectation and variance for the number of table reservations where the customers turn up. [2]

**(ii)** Find the probability that at least 9 table reservations but less than 13 have their customers turn up. [3]

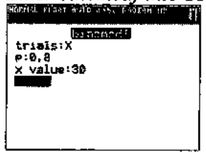
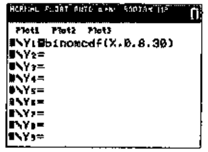
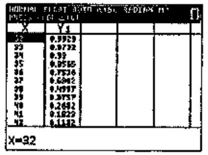
**(b)** Find the probability that, for dinner service on two days with 14 table reservations each, there is a total of exactly 24 table reservations where the customers turn up. [2]

**(c)** A restaurant manager of a 30-table restaurant decides to offer more table reservations than the full capacity. Find the maximum number of table reservations that the manager can offer such that there is a probability of at least 85% that the restaurant will not exceed full capacity for a dinner service. [3]

| 8 | Solution [10] Binomial Distribution | Comments |
|---|---|---|
| **(ai)** | Let $X$ be the random variable denoting the number table reservations where the customer did turn up for the dinner out of 14 table reservations.<br>$X \sim B(14, 0.8)$<br>$E(X) = (14)(0.8)$<br>$= 11.2$<br>$\text{Var}(X) = (14)(0.8)(0.2)$<br>$= 2.24$ | Most can do this quite easily. Take note however that 80% means just 0.8 and not 0.80 strictly speaking.<br><br>And here, the context is all about **NUMBER OF TABLE RESERVATIONS** and not number of customers. |
| **(aii)** | $P(9 \le X < 13) = P(9 \le X \le 12)$<br>$= P(X \le 12) - P(X \le 8)$<br>$= 0.75823$<br>$= 0.758$ (3 s.f) | Poorly attempted despite this being a very simple part to score in.<br>Quite many somehow interpreted this as a conditional probability situation which is wrong.<br>A handful of students went on to treat as normal distribution, most likely having **mistakenly assumed** that expectation and variance information implies normal distribution (very serious conceptual error). |
| **(b)** | Let $Y$ be the random variable denoting the number table reservations where the customer did turn up for the dinner out of 28 table reservation.<br>$Y \sim B(28, 0.8)$<br>$P(Y = 24) = 0.155$ (3 s.f) | Here, you need to combine the scenario for 2 days so total number is now out of 28. Probability is still 0.8.<br><br>A small number keyed in GC wrongly by using binomcdf, which is wrong. You are required to use binompdf here! |





| | | |
|---|---|---|
| **(c)** | Let $T$ be the random variable denoting the number of table reservations where the customers turn up, out of $n$ table reservations.<br>$T \sim B(n, 0.8)$<br>$P(T \le 30) \ge 0.85$<br>From GC,<br><br>$n$ \| $P(T \le 30)$<br>35 \| $0.85651 \ge 0.85$<br>36 \| $0.75363 < 0.85$<br><br>Maximum $n = 35$<br><br>*Guidance screenshots for GC keystrokes:*<br>*Enter binomcdf into Y1*<br><br><br>*(see when it last reaches more than 0.85)* | More well answered than (a)(ii) which is surprising as this is considered a more challenging part.<br><br>Those who could not get it sometimes confuse the notations.<br><br>Take note the issue to solve here is $n$ (represented as unknown X in GC).<br>Full capacity means at most 30.<br><br>Here, the answer must be shown with a relevant table as shown on the left to be awarded the full marks. |
| | | Students are strongly recommended to compare this question with the 2022 prelim P2/Q11 paper on the left handed characteristic scenario and see the difference. It would complete your learning more meaningfully. |



BP-613

9 **(a)** A scientist is studying the growth of water lilies in a large lake. He planted a water lily at one corner and he measures the area, $A$ km$^2$, the water lilies cover on day $t$. His results are recorded below:

| Time, $t$ (days) | 1 | 4 | 7 | 14 | 20 | 29 |
|---|---|---|---|---|---|---|
| Area, $A$ (km$^2$) | 0.6 | 1.3 | 3.7 | 7.4 | 8.6 | 9.2 |

**(i)** Draw a scatter diagram to illustrate the data. [1]

**(ii)** The scientist would like to predict the future growth of the water lilies. Using the scatter diagram and the context of the question, state two reasons why, in this context, a linear model is not appropriate. [2]

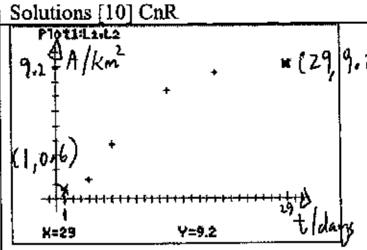
**(b)** It is proposed to fit the above data with a model of the form $\ln(D-A)=a+bt$, where $D$ is a suitable constant. The product moment correlation coefficient between $t$ and $\ln(D-A)$ is denoted by $r$. The following table gives values of $r$ for some possible values of $D$.

| $D$ | 9.5 | 9.8 | 10 |
|---|---|---|---|
| $r$ | | −0.99359 | −0.99114 |

**(i)** Calculate the value of $r$ for $D=9.5$, giving your answer correct to 5 decimal places. Hence, explain which of 9.5, 9.8 or 10 is the most appropriate value of $D$ for the model to fit. [2]

**(ii)** Using this value of $D$, calculate the values of $a$ and $b$ correct to 5 decimal places, and use them to predict the area covered by water lilies after 28 days. Comment on whether the estimate is reliable. [4]

**(iii)** Give an interpretation, in context, of the value of $D$. [1]

| 9 | Solutions [10] CnR | Comments |
|---|---|---|
| (ai) |  | Common errors:<br>• Not indicating units for axes<br>• Not labelling range of values of $A$ and $t$ |
| (aii) | A linear model is not likely to be appropriate as the area covered would then increase infinitely. However, the area of lake is finite.<br><br>Moreover, the scatter diagram shows a curvilinear trend such that as $t$ increases, $A$ increases at a decreasing rate, which are not well represented by linear model. | Some students provided reasons just based on scatter diagram but not on the context of the question.<br>For the contextual reason, students should highlight the implication of linear model in that the area covered would increase infinitely and that it is not possible due to limit in size of lake. |
| (bi) | For $D=9.5$, $r=-0.99602$<br>Since $D=9.5$ gives a value of $\lvert r\rvert$ <u>is closer to 1</u>, compared to the other 2 values, it is the most appropriate for the model to fit. | Some students had difficulties in correct set up of GC data entry in calculation of $r$ value and subsequently the value of $a$ and $b$ for part b(ii).<br>Some students provided wrong reason stating that $r$ is closer to 1 instead of $r$ closer to −1 or $\lvert r\rvert$ is closer to 1. |
| (bii) | $a=2.510165315$; $b=-0.1277379$<br><br>Equation of regression line is<br>$\ln(9.5-A)=2.51017-0.12774t$<br><br>When $t=28$, $\ln(9.5-A)=-1.06655$<br>$\ln(9.5-A)=-1.06655$<br>$9.5-A=0.3441939$<br>$A=9.15581$<br><br>When $t=28$, it is <u>within the data range ($1\le t\le 29$)</u>, and since the <u>product moment correlation coefficient between $\ln(9.5-A)$ and $t$ is close to −1</u>, the estimate calculated for $A$ is expected to be reliable. | Generally no problem for this part except for students not leaving their answers for $a$ and $b$ in 5 decimal places.<br><br>Many students did not mention about value of $r$ being close to −1 in concluding that estimate is reliable. |
| (biii) | $D$ is the likely long-term maximum/upper bound area covered by the water lilies.<br><br>(Note: It is the maximum as<br>$\ln(D-A)=2.51-0.128t \Rightarrow D-A>0 \Rightarrow A<D$.<br>Moreover,<br>$\ln(D-A)=2.51-0.128t$<br>$\Rightarrow D-A=e^{2.51-0.128t} \Rightarrow A=D-e^{2.51-0.128t}$.<br>When $t\to\infty$, $e^{2.51-0.128t}\to 0$.<br>This implies that when $t\to\infty$, $A\to D$.) | Common errors:<br>• $D$ is the maximum area of the lake<br>• $D$ is the total area of the lake<br>For students who mentioned about area covered by the water lilies, they failed to highlight about $D$ being the likely long-term maximum value. |

**10** An internet advertising company *TicTakAim* claims that viral video's duration has a mean duration of 30 seconds. An influencer wants to investigate the company's claim as she believes that the company is underestimating the mean duration. However, she is unable to record the durations of all the viral videos.

**(a)** Explain how she could obtain a sample of viral video durations, and why she should obtain the sample in this way. [2]

The influencer takes a sample of 90 viral video's durations. The viral video's durations, $x$ seconds, are summarised as follows.

$$\Sigma(x-30)=90 \qquad \Sigma(x-30)^2=2037$$

**(b)** Find the unbiased estimates of the population mean and variance of the durations of viral videos. [2]

**(c)** Carry out an appropriate test, at the 3% level of significance, whether the company's claim is justifiable. You should state your hypotheses and define any symbols you use. [4]

**(d)** Explain, in the context of the question, the meaning of "at the 3% level of significance". [1]

**(e)** The influencer was later informed that the population standard deviation of the viral video duration is $\sigma$ seconds. Find the set of values of $\sigma$ so that the influencer can conclude that there is sufficient evidence at the 3% level of significance to believe that *TicTakAim* is underestimating the mean duration. [3]

| 10 | Solution [12] Hypothesis Testing | Comments |
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| (a) | She should conduct a **random sampling** of the population of the duration of the viral videos.<br>This is because, the sampling will be **unbiased** or that each viral video will have an equal chance of being selected into the sample. | Most responses stated that a random sample should be chosen; quite a number gave details of how this could be done, which was not required and was not always random. In this question, explicit knowledge on how to obtain the sample is not required.<br><br>There is a need to explain that the reason for random sampling – **to avoid bias**. Others brought up the reason as "each video will have an equal chance of being selected into the sample" but often neglected the phrase "into the sample" or used other words like "same", "constant". Students are advised to stick to standard key words and phrasings.<br><br>Some students mentioned |





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| | | only that duration of viral videos should be chosen independently without also saying that they should have an equal chance of being chosen, which is insufficient.<br><br>A significant majority of responses included that the sample size should be of at least 30 so the Central Limit Theorem can be used. However, there is **no need to do so** in this part as the question has not specified that a hypothesis test is to be conducted. |
| (b) | Unbiased estimate of population mean, $\bar{x}=\frac{90}{90}+30=31$<br><br>Unbiased estimate of population variance,<br>$s^2=\frac{1}{89}\left[2037-\frac{90^2}{90}\right]=\frac{1947}{89}=21.876=21.9\,(3\text{ s.f.})$ | Generally well done.<br><br>Except for some making very careless mistakes in calculations or mistakenly applying the incorrect formula for $s^2$ |
| (c) | Let $X$ denote the mean duration of the viral videos in seconds and $\mu$ be the population mean duration of the viral videos in seconds.<br>Test $H_0: \mu=30$<br>against $H_1: \mu>30$<br>at 3% level of significance.<br>Test statistic:<br>Under $H_0$, as $n=90\geq 30$ is large, by CLT,<br>$\bar{X}\sim N\left(30,\frac{s^2}{90}\right)$ approximately ($s^2=21.876$.)<br>$Z=\frac{\bar{X}-30}{s/\sqrt{90}}\sim N(0,1)$ approximately<br>$p$-value $=0.021264<0.03$,<br>(OR $z_{calc}=2.0283>z_{critical}=1.8808$)<br>Thus, we reject $H_0$.<br>There is **sufficient evidence** at **3% level of significance** that the **mean duration is more than 30 seconds.** i.e. the company's has underestimated the duration. | Generally majority of students are able to carry out the procedures of hypothesis testing. However, a significant number are still lacking in proper presentations.<br><br>Common lapses include:<br>- Not defining $\mu$<br>- Using $\mu_0$ and $\mu_1$ in place of $\mu$ in the hypotheses<br>- $\bar{X}\sim N\left(31,\frac{s^2}{90}\right)$<br>- $Z=\frac{31-30}{s/\sqrt{90}}\sim N(0,1)$<br>- Incorrect/Incomplete phrasing of conclusion |



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| (d) | "At the 3% significance level" means that there is a **probability of 0.03** that the test concludes that the **mean duration is greater than 30 seconds** when **in fact it is 30 seconds**.<br><br>OR There is a probability of 0.03 that the test concludes that the company underestimated the duration when in fact it did not. | A significant of students could not remember this definition and invented their own definitions which were incorrect.<br>Also, it is not sufficient to define significance level as the "Probability of rejecting $H_0$ when $H_0$ is true", there is a need to **contextualize** the definition. |
| (e) | Test $H_0: \mu = 30$<br>against $H_1: \mu > 30$<br>at 3% level of significance.<br>Test statistic:<br>Under $H_0$, as $n = 90 \geq 30$ is large, by CLT,<br>$\bar{X} \sim N\left(30, \frac{\sigma^2}{90}\right)$ approximately<br>$Z = \frac{\bar{X} - 30}{\sigma/\sqrt{90}} \sim N(0,1)$ approximately<br>Critical region: $z > z_{critical} = 1.8808$<br>Since $H_0$ is rejected, $z_{calc}$ is in the critical region.<br>$z_{calc} > 1.8808$<br>$\frac{31-30}{\sigma/\sqrt{90}} > 1.8808$<br>$\sigma < 5.04404$<br>$\sigma < 5.04$ (3 s.f.)<br>$\{\sigma \in \mathbb{R} : 0 \leq \sigma < 5.04\}$. | There was a varying performance for this part.<br><br>However, even for those who succeeded in obtaining the solution, students commonly had lapses in setting up the test, for example:<br>- $X \sim N(30, \sigma^2)$<br>- $Z = \frac{31-30}{\sigma/\sqrt{90}} \sim N(0,1)$<br>$Z = \frac{\bar{X}-31}{s/\sqrt{90}} \sim N(0,1)$<br>- $Z = \frac{\bar{X}-30}{\sigma} \sim N(0,1)$<br>For those who did not succeed, common errors include:<br>- Mistakenly finding variance by taking $\frac{90}{89}\left(\frac{\sigma^2}{90}\right)$<br>- $z_{calc} = \frac{30-31}{\sigma/\sqrt{90}}$<br>- Critical regions used for right tail or two tail test<br>Note: By convention when rejecting $H_0$, we do not include the equal sign i.e. $z_{calc} > 1.8808$ instead of $z_{calc} \geq 1.8808$. However students are not penalized for including the equal sign. |
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**11** A leather craftsman customized leather belts according to the widths of the customer's buckles. Over a period of time, it is found that the buckle widths are normally distributed. 60% of the buckles have width more than 25 mm and 15% are less than 24 mm.

**(a)** Find the mean and variance of the buckle width. [3]

The widths of the leather belts produced by the craftsman follow a normal distribution with mean 25.1 mm and standard deviation 1.4 mm.

**(b)** Find the probability that the width of a randomly chosen leather belt is between 24 mm and 26 mm. [1]

In order to fit the leather belts nicely into the buckles, the craftsman reduces the widths of these leather belts by 1%.

**(c)** Find the probability that the total width of 3 randomly chosen leather belts is less than 75.4 mm. [3]

There are holes that are punctured into the leather belts that have diameters, in mm, that follow the distribution $N(4.5, 0.2^2)$.

The prong is part of the belt buckle that is also known as the pin or the "fork". It goes through any of the holes in the belt to secure the belt in place.
The diameter of the prong, in mm, follows the distribution $N(4.3, 0.1^2)$.

If the diameter of a prong is more than 0.2 mm greater than the diameter of a hole, then the hole has to be enlarged to make it fit.
If the diameter of a hole is more than 0.3 mm greater than the diameter of a prong, welding is done to increase the diameter of the prong to make it fit.

**(d)** A complete set of a belt is made up of a randomly chosen buckle with a prong and a leather belt with 5 punctured holes. Find the probability that for a belt, the prong can be fitted into every hole without having the holes enlarged or the prong welded. [4]

**(e)** A punctured hole on a belt and a buckle with a prong are randomly chosen for inspection. State with a reason whether or not the event that the hole needs to be enlarged and the event that the prong needs to be welded are independent. [2]

| 11 | Solutions [13] Normal Distribution | Comments |
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| (a) | Let $X$ be the r.v. that denotes the width of the buckle.<br>$X \sim N(\mu, \sigma^2)$<br>$P(X > 25) = 0.6$<br>$P\left(Z > \frac{25-\mu}{\sigma}\right) = 0.6$<br>$\frac{25-\mu}{\sigma} = -0.25335$<br>$\mu - 0.25335\sigma = 25$ | It was shocking that this question proved challenging to many students.<br><br>This question type is classic and is very common in A levels, students are recommended to master this question.<br><br>Many did not attempt the question.<br><br>Many did not have appropriate strategies to solve this question. Many thought it was |

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| | $P(X<24)=0.15$<br>$P\left(Z<\frac{24-\mu}{\sigma}\right)=0.15$<br>$\frac{24-\mu}{\sigma}=-1.0364$<br>$\mu-1.0364\sigma=24$<br>Solving,<br>$\mu=25.3$ and $\sigma=1.2771$, $\sigma^2=1.63$ (3 s.f.) | DRV and attempted to use formulas based on it.<br><br>Students **incorrectly** went to find the mean via $\frac{24+25}{2}$, but this cannot be the case as the probabilities give are not equal, i.e. they are not symmetrical about the mean.<br><br>Stronger responses went to standardise the normal distribution appropriately and use 'InvNorm' to deduce the value of the limits. Then, they solved the simultaneous equation (quickiest was using the GC) to deduce the mean and standard deviation. Lastly, they squared the s.d. to find the variance. |
| **(b)** | Let $Y$ be the r.v. that denotes the width of the leather belt.<br>$Y \sim N(25.1, 1.4^2)$<br><br>$P(24<Y<26)=0.524$ (3 s.f.) | Though well done, it is a good reminder to define your variables. |
| **(c)** | Let $B$ be the r.v. that denotes the **reduced** width of the leather belt.<br>$B=0.99Y$<br>$B \sim N(25.1\times0.99, 1.4^2\times0.99^2)$<br>$B \sim N(24.849, 1.386^2)$<br><br>$B_1+B_2+B_3 \sim N(74.547, 5.762988)$<br>$P(B_1+B_2+B_3<75.4)=0.63884$<br>$=0.639$ (3 s.f.) | Generally well done.<br><br>There is a minority of students that multiplied 0.99 to the mean **but not the variance**. Please recall that $Var(kX)=k^2Var(X)$.<br><br>Another minority took $B_1+...+B_3$ as $3B$ instead.<br><br>There is another minority that did not press their GC correctly, inputting the variance instead of the standard deviation to calculate the probability. |
| **(d)** | Let $H$ and $P$ be the r.v. that denotes the diameter of the hole and prong, in mm respectively.<br>(Assume that the difference between diameter of the prong and one hole is independent to that the difference between diameter of the prong and another hole.) | This question proved challenging to students.<br><br>Many students saw $H-P$ and $P-H$ as different |





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| | $H \sim N(4.5, 0.2^2)$<br>$P \sim N(4.3, 0.1^2)$<br>$H-P \sim N(0.2, 0.05)$<br><br>$H-P \sim N(0.2, 0.05)$<br>$P(\text{to enlarge hole})=P(H-P<-0.2)=0.036819$<br>$P(\text{to weld pronge})=P(H-P>0.3)=0.32726$<br><br>Required Prob<br>$=[1-P(H-P<-0.2)-P(H-P>0.3)]^5$<br>$=0.103914$<br>$=0.104$ (3 s.f.)<br><br><u>**Alternatively**</u><br>Required Prob<br>$=[P(-0.2\le H-P\le 0.3)]^5$<br>$=0.103914$<br>$=0.104$ (3 s.f.)<br><br>(Note: If independence is not assumed, we will have to engage in a double integral. We consider the when the diameter of the prong is of a diameter $x$ then find the probability when the diameter of the hole satisfy the condition:<br>Required Prob<br>$=\int_{-\infty}^{\infty}\frac{1}{0.1\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-4.3}{0.1}\right)^2}[P(-0.2\le H-x\le 0.3)]^5\,dx$<br>$=0.144$ (3 s.f.)<br>) | independent variables, but they are just negatives of each other! This resulted in $P(-0.2\le H-P\le 0.3)$ being **incorrectly** written as $P(H-P\le 0.3)\times P(P-H\ge 0.2)$.<br><br>Students also **incorrectly** resorted to finding the sum or mean of 5 holes or multiplying 5 to deal with 5 holes. Instead, one should have powered 5, as we are multiplying the same probability by itself for 5 times.<br><br>A minority of students went to deduct the variances instead of adding them as well. |
| **(e)** | Now,<br>$P(\{H-P<-0.2\}\cap\{H-P>0.3\})=0$<br><br>$P(H-P<-0.2)P(H-P>0.3)=0.32726\times0.036819$<br>$=0.01205\ne 0$<br>(Or that $P(H-P<-0.2), P(H-P>0.3)>0$)<br>Therefore,<br>$P(H-P<-0.2)P(H-P>0.3)$<br>$\ne P(H-P<-0.2\cap H-P>0.3)$<br>Hence, the event that the hole needs to be enlarged is **not** independent from the event that the prong needs to be weld. | This question proved extremely challenging to most students.<br><br>Many students went at length on why one variable might 'affect' the other, providing iffy and imprecise reasoning. Some were confused between the independence of the diameter of the prong and hole and the independence of the events of enlargement and welding. Many has completely forgotten the |



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| | | mathematical definition of independence.<br><br>Better attempts show that the events are mutually exclusive, but failed to provide reason why this implied that the events were not independent. There were some attempts that confused 'mutually exclusive' events with 'independent' events.<br><br>The best responses directly **went head on to the definition of independence** and showed that the mutually exclusive events cannot be independent. |
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**END OF PAPER**