TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ______________________

CIVICS GROUP: ______________________

H2 MATHEMATICS

9758/01

Paper 1

10 SEPTEMBER 2024

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
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Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

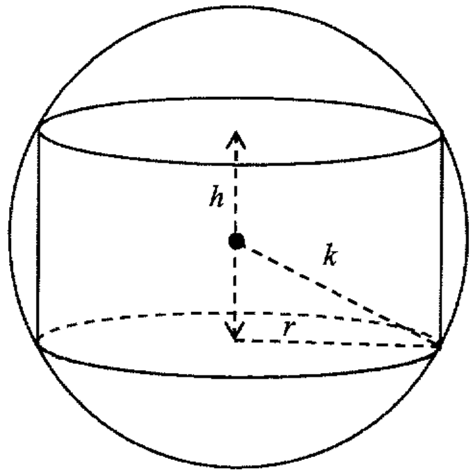
This document consists of 25 printed pages and 1 blank page.

[Turn Over

1 (i) Express $y = \frac{2x+1}{x-4}$ in the form $y = a + \frac{b}{x-4}$ where a and b are constants to be determined. [1]

(ii) Hence, state a sequence of transformations that will transform the curve with equation $y = \frac{1}{x}$ to the curve with equation $y = \frac{2x+1}{x-4}$. [3]

2



A cylinder with height h cm and base radius r cm is inscribed within a sphere with fixed radius k cm. The circumference of the circular top and bottom of the cylinder is in contact with the sphere (see figure above). By using differentiation, find, in terms of k, the exact value of h for which the volume of the cylinder is maximum. [6]

3 The function f is defined by

$$f(x) = \begin{cases} 2\sin\left(\frac{\pi x}{4}\right) & \text{for } 0 \le x < 2, \\ 6 - 2x & \text{for } 2 \le x < 3, \end{cases}$$

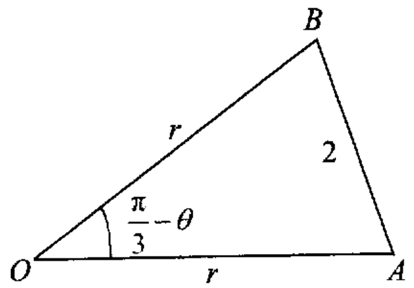
and that $f(x) = f(x+3)$ for all real values of x.

(i) Find the value of $f(7)$. [1]

(ii) Sketch the graph of $y = f(x)$ for $-4 \le x \le 7$. [3]

(iii) The region R is bounded by the curve $y = f(x)$ and the x-axis from $x = 0$ to $x = 5$. Find the exact volume of solid generated when R is rotated 2π radians about x-axis. [4]

4



In the diagram above, OAB is an isosceles triangle where $OA = OB = r$ cm. It is given that the length $AB = 2$ cm and angle $AOB = \frac{\pi}{3} - \theta$ radians.

(i) Using cosine rule, show that $r^2 = \dfrac{4}{2 - \sqrt{3}\sin\theta - \cos\theta}$. [3]

(ii) Given that θ is a sufficiently small angle, show that

$$r \approx 2 + a\theta + b\theta^2$$

where a and b are constants to be determined. [4]

5 An arithmetic progression A has first term a and common difference d, where a and d are non-zero. A geometric progression G has first term b and common ratio r.

(i) The first, third and eleventh terms of A are equal to the fourth, third and second term of G respectively. Prove that the geometric series of G is convergent. [4]

(ii) It is given instead that $r = \frac{1}{3}$ and the terms of another sequence H is formed by squaring the terms of G. Find the range of values of b such that the sum to infinity of H exceeds the sum to infinity of G by more than $\frac{3}{2}$. [3]

6 It is given that $f(r) = \frac{1}{2^r + 1}$.

(i) Show that $f(r) - f(r+1) = \frac{2^r}{(2^r + 1)(2^{r+1} + 1)}$. [1]

(ii) Using the result in part (i), find $\sum_{r=1}^{n} \frac{2^r}{(2^r + 1)(2^{r+1} + 1)}$. [3]

(iii) Hence, find $\sum_{r=1}^{n}\left[\frac{2^{r+3}}{\left(2^{r+2}+1\right)\left(2^{r+3}+1\right)}-(2r+5)\right]$. [4]

7 A curve C has equation $y^2 = 3x^3 - x + 1$.

(i) Find the exact coordinates of all the stationary points. [4]

(ii) A particle moves along C to the stationary point S, found in part **(i)** where both the x-coordinate and y-coordinate are positive. Given that its x-coordinate is increasing at a rate of $\frac{1}{9}$ units per second, find the exact rate of change of the gradient of C when the particle is at point S. [4]

8 Figure 1 shows a triangular prism *OABCDE*, where *O* is the origin. Point *F* lies on *AE* such that *AF*:*FE* = $1-k:k$ where $0<k<1$. Figure 2 shows a model of a crystal trophy that is made by cutting away a tetrahedral section *CDEF* from the triangular prism *OABCDE*. With respect to *O*, the coordinates of *A*, *B* and *D* are $(4,3,0)$, $(0,6,0)$ and $(0,0,20)$ respectively. It is given that $\overrightarrow{OD}=\overrightarrow{BC}=\overrightarrow{AE}$.

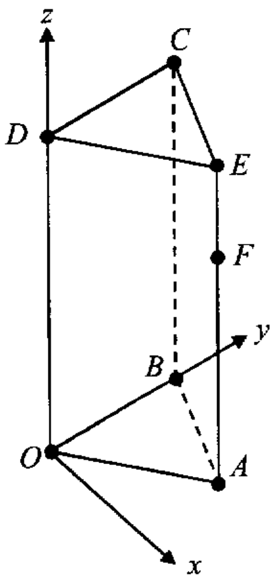


Figure 1

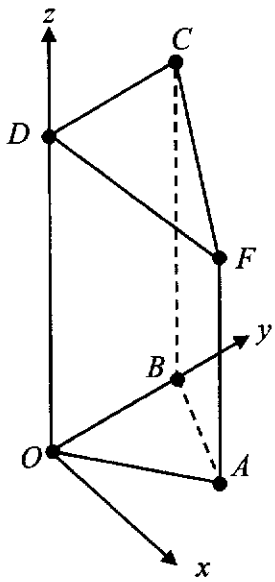


Figure 2

(i) Write down the position vector of *E* and find the position vector of *F* in terms of *k*.

[2]

(ii) Show that the equation of the plane CDF can be written as $\mathbf{r}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20.$ [3]

(iii) Find the x-coordinate of the foot of perpendicular from the point E to the plane CDF in terms of k. [3]

(iv) Hence, find the value of k such that the reflection of the plane CDE in the plane CDF lies on the plane $OBCD$. [4]

9 The function f is defined as follows:

$$f : x \mapsto \frac{1}{(x-2)^2} \quad \text{for} \quad x \in \mathbb{R}, x > k.$$

(i) State the least value of k for which the function f^{-1} exists. [1]

For the rest of the question, take the value of *k* as the value found in part (i).

(ii) Sketch, on the same diagram, the graph of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the relationship between the two graphs. [3]

(iii) Find the exact value of $f^{-1}(5)$. [2]

The function g is defined as follows:

$$g : x \mapsto \frac{1}{\sqrt{x}} - 4\sqrt{x} \quad \text{for} \quad x \in \mathbb{R}, x > 0.$$

(iv) Show that gf^{-1} exists and find its range in exact form. [3]

(v) Solve the inequality $gf(x) \leq 0$ algebraically. [4]

10 A company proposed to build a water detention tank to address the flooding problem in a village. To test the feasibility of the proposal, the company created a model of a water detention tank, with a capacity of 300 cm^3.

Water is pumped into the model of the tank at a constant rate of 50 cm^3/min and pumped out at a rate proportional to the volume of water in the tank. At time t minutes, the volume of water in the model of the tank is V cm^3.

(i) Write down a differential equation for this situation. [1]

(ii) Solve this differential equation to obtain the general solution for V in terms of t. [4]

Initially, the volume of water in the tank is 100 cm^3. After a minute, the volume increased to 130 cm^3.

(iii) Hence, show that the particular solution for V in terms of t is $V = 289 - 189e^{-0.173t}$, correct to 3 significant figures. [4]

(iv) Find how long it will take for the tank to reach 95% of its maximum capacity. [1]

(v) Sketch the graph of V against t. [2]

(vi) Explain, with justification, what will happen to the volume of water in the tank in the long run. [1]

11 Mechanical engineers are responsible for the design of the drops and loops in roller coaster tracks. In order to design a roller coaster ride that is exciting, yet safe, mechanical engineers are required to possess a strong understanding of force, gravity, motion, momentum, and potential and kinetic energy.

A mechanical engineer first designs part of a roller coaster track using the curve C, which is defined by the parametric equations

$$x = t^2 + 2t,\ y = \ln(t+1) \text{ where } t > -1.$$

(i) Sketch the part of the roller coaster track that is defined by C. [2]

The mechanical engineer then designs a drop in the roller coaster track. The drop is modelled after the equation of the line N, which is the normal to C at the point where $t=2$.

(ii) Find the exact equation of line N. [3]

(iii) For the roller coaster drop to be deemed safe, the acute angle between line N and the x-axis needs to be between $30°$ and $80°$. Comment on the suitability of using line N in the design of the roller coaster drop. [1]

An advertising billboard is designed to be mounted on the rollercoaster track. Its area can be modelled by the area enclosed by curve C, line N and the x-axis.

(iv) Show that the area of the billboard is $a(\ln 3)^2 + b\ln 3 + c$, where a, b and c are constants to be determined. [8]

End of Paper

TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ______________________________

CIVICS GROUP: ______________________________

H2 MATHEMATICS

9758/02

Paper 2

16 SEPTEMBER 2024

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

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The total number of marks for this paper is 100.

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Section A: Pure Mathematics [40 marks]

1 **(i)** Given that $y=\ln\left(1+3x+2x^2\right)$, show that $\left(1+3x+2x^2\right)\frac{d^2y}{dx^2}+\left(3+4x\right)\frac{dy}{dx}=4$.

By further differentiating the above result, find the Maclaurin series for y, up to and including the term in x^3. [5]

(ii) Verify the correctness of your answer in part **(i)** by using the standard results given in the List of Formulae (MF26). [2]

(iii) Hence, by using $x = \frac{1}{2}$, estimate the value of $\ln 3$. [2]

2 Referred to the origin O, the points A and B have position vectors **a** and **b** respectively. The point C is such that O divides the line segment AC in the ratio $2:3$. The point D divides the line segment AB in the ratio $1:4$.

(i) Show that the area of triangle OCD is given by $\lambda|\mathbf{a}\times\mathbf{b}|$, where λ is a constant to be determined. [5]

(ii) It is now given that **a** is perpendicular to **b**. Given that $\overrightarrow{AP}$ is the projection vector of $\overrightarrow{AO}$ onto $\overrightarrow{AB}$, show that $\overrightarrow{AP} = \mu(\mathbf{b} - \mathbf{a})$, where μ is a constant to be determined in terms of $|\mathbf{a}|$ and $|\mathbf{b}|$. [3]

3 **Do not use a graphing calculator when answering this question.**

(a) The equation $z^3 - 8z^2 + 23z + k = 0$, where k is a real constant, has a root $2 + \sqrt{3}\text{i}$.

(i) Find the value of k and solve the equation. [5]

(ii) Hence, solve the equation $z^3 + 8\text{i}z^2 - 23z + k\text{i} = 0$ [2]

(b) The complex number w is given by $w = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \frac{\pi}{2}$. Find, in terms of θ, the modulus and argument of $\frac{w^*}{w^2+1}$, where w^* is the complex conjugate of w. [4]

4 (a) (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\cos x}\right)$. [1]

(ii) Hence, show that

$$\int \sin 2x \, \mathrm{e}^{\cos x} \, \mathrm{d}x = 2\mathrm{e}^{\cos x}(1-\cos x)+C,$$

where C is an arbitrary constant. [3]

(iii) Find the exact value of $\int_0^{\pi} \left|\sin 2x \, \mathrm{e}^{\cos x}\right| \mathrm{d}x$. [3]

(b) Using the substitution $x = \sec\theta$, where $0 < \theta < \frac{\pi}{2}$, find $\int \frac{1}{\sqrt{x^2 - 1}}\,dx$. [5]

Section B: Probability and Statistics [60 marks]

5 For events A and B, it is given that $\mathrm{P}(A)=\frac{1}{8}$, $\mathrm{P}(A\cup B)=\frac{9}{16}$ and $\mathrm{P}(A\cap B')=\frac{1}{16}$.

(i) Find $\mathrm{P}(B)$. [1]

(ii) Determine, with a reason, whether the events A and B are independent. [2]

For a third event C, it is given that $\mathrm{P}(C)=\frac{3}{8}$, $\mathrm{P}(A\cap B\cap C)=\frac{1}{64}$ and C is independent of A.

(iii) Find $\mathrm{P}(A'\cap C)$. [1]

(iv) Find the range of values for $P(A' \cap B' \cap C')$. [3]

6 At a carnival, a game stall offers a game of throwing darts. The darts will land either in the bull's eye, inner ring or outer ring. There is a probability of p, $2p$ and $5p$ of landing in the bull's eye, inner ring and outer ring respectively.
A player gets five points if the dart lands in the bull's eye, three points if the dart lands in the inner ring and one point if the dart lands in the outer ring. A round consists of two throws. Let X be the total number of points for one round of two throws.

(i) Find the value of p and show that $\mathrm{P}(X=6)=\frac{7}{32}$. Obtain the probability distribution of X. [5]

(ii) Find $\mathrm{E}(X)$. [1]

(iii) A player wins \$1 for every two points earned. If a player pays \$$k$ to play one round of two throws, find the value of k for the game to be fair. [2]

7 The owner of a popsicles store would like to study the effect of daily average temperature (x) on the number of popsicles sold (y). A random sample of 10 days is taken and the results are shown in the table.

Average Temperature, $x(°C)$	22	23	24	26	27	28	29	31	33	36
Number of popsicles sold (y)	90	102	118	145	134	159	166	179	185	190

(i) Draw a scatter diagram for these values, labelling the axes clearly. Comment on the relationship between x and y. [3]

(ii) By calculating the product moment correlation coefficients, determine and explain whether the relationship between x and y is modelled better by $y = a + bx$ or $\ln(200 - y) = c + dx$. Find the equation of the suitable regression line. [4]

(iii) The store owner would like to estimate the number of popsicles sold when the daily average temperature reaches $37\,^{\circ}C$. Find the estimate and explain whether the estimate is reliable. [2]

(iv) Give a possible interpretation, in context, of the value 200 in the model $\ln(200-y)=c+dx$. [1]

8 **In this question you should state the parameters of any normal distribution(s) that you use.**

A fruit seller claims that his oranges have a mean mass of 365g. A consumer association would like to test whether the fruit seller has overstated the mean. To conduct this test, a random sample of 40 oranges is taken and the mean mass is found to be 364.2g and has standard deviation 4g.

(i) Test, at the 5% level of significance, whether the fruit seller has overstated the mean. [5]

The diameter of oranges in centimetres is a normally distributed continuous random variable D. The standard deviation of D is 0.3 cm and under ordinary conditions the expected value of D is 9 cm. A test is carried out, at the 5% significance level, to determine whether the mean diameter exceeds 9 cm.

(ii) Given that the sample size is 50, find the range of values within which the mean diameter of this sample must lie, such that there is sufficient evidence from the sample to conclude that the mean diameter exceeds 9 cm. [4]

(iii) Another sample of fifty oranges is taken and the mean diameter of the sample is found to be 8.9 cm. Without any further calculations, explain, with justification, the conclusion that can be made at the 5% significance level. [1]

9 A carnival game involves one round of tossing 5 balls into a container, one at a time. It is known that, on average, 35% of the balls tossed will go into the container. The participant will win a prize if he tosses at least 3 balls into the container. The number of balls that a participant tosses into the container in one round is denoted by X.

(i) State, in the context of the question, two assumptions needed to model X by a binomial distribution. Explain why one of the assumptions may not hold. [3]

You are now given that X can be modelled by a binomial distribution.

(ii) Find the probability that a randomly chosen participant wins a prize. [2]

Ten randomly chosen participants played the carnival game once.

(iii) Find the probability that at most 6 of them did not win a prize. [2]

The game is modified to allow more participants to win a prize, where participants may play one or two rounds of the game.

- If the participant tosses at least 3 balls into the container in the first round, he wins a prize.
- If the participant tosses fewer than 2 balls into the container in the first round, he will not win any prize.
- If the participant tosses exactly 2 balls into the container in the first round, he will play a second round. If he tosses at least 4 balls into the container in the second round, he wins a prize.

(iv) Find the probability that a randomly chosen participant wins a prize in this modified game. [2]

(v) Given that a randomly chosen participant wins a prize in a modified game, find the probability that he tossed fewer than 7 balls into the container. [3]

10 In this question you should state clearly the values of the parameters of any normal distribution(s) that you use.

Papayas and watermelons are sold by weight. The masses, in kg, of papayas and watermelons are modelled as having normal distributions with means and standard deviations as shown in the table.

	Mean mass	Standard deviation
Papayas	1.85	0.12
Watermelons	6.50	0.72

(i) Sketch the distribution for the masses of papayas between 1.4 kg and 2.3 kg. [2]

(ii) Find the probability that the mass of a randomly chosen papaya is less than 1.7 kg. [1]

(iii) Find the probability that the mass of a randomly chosen watermelon exceeds the mass of a randomly chosen papaya by more than 4.5 kg. [3]

(iv) The mean mass of n randomly chosen watermelons is denoted by $\overline{Y}$ kg. Given that $P(\overline{Y} > k) = 0.2$, express k in terms of n. [3]

Papayas are sold at \$2.20 per kg and watermelons at \$1.45 per kg.

(v) Calculate the probability that the total selling price of 3 randomly chosen papayas and 4 randomly chosen watermelons exceed \$50. [3]

(vi) State an assumption for your calculations in parts **(iii)**, **(iv)** and **(v)**. [1]

End of Paper

BP-733

Qn	Solution
1	**Transformation of Curves**
(i)	$y=\dfrac{2x+1}{x-4}=\dfrac{2(x-4)+9}{x-4}=2+\dfrac{9}{x-4}$ where $a=2$ and $b=9$.
(ii)	**Note: For sequence of transformation questions, you MUST describe the transformations (using the keywords) and not just write the replacements**

Template {delete as appropriate}:

Translate ___ units in the {positive / negative} {x-direction / y-direction}	
x-direction [Replace x by $x-(k)$]	y-direction [Replace y by $y-(k)$]
• **Sign of** (k) **determines positive / negative** • **Magnitude of** (k) **determines no. of units of translation**	
Stretch by a factor of k parallel to the {x-axis / y-axis}	
Parallel to the x-axis $\left[\text{Replace } x \text{ by } \dfrac{x}{k}\right]$	Parallel to the y-axis $\left[\text{Replace } y \text{ by } \dfrac{y}{k}\right]$
• k is the stretch factor	
Reflection in the {x-axis / y-axis}	
In the y-axis [Replace x by $-x$]	In the x-axis [Replace y by $-y$]

Method 1: Stretch parallel to y-axis

$$y=\frac{1}{x}\xrightarrow{(1)}y=\frac{1}{(x-4)}\xrightarrow{(2)}\left(\frac{y}{9}\right)=\frac{1}{x-4}\Rightarrow y=\frac{9}{x-4}\xrightarrow{(3)}y=2+\frac{9}{x-4}$$

(1) Translate 4 units in the positive x-direction
(2) Stretch by a factor of 9 parallel to the y-axis
(3) Translate 2 units in the positive y-direction

Alternative: In the sequence (2), (3), (1) **or** (2), (1), (3)

Method 2: Stretch parallel to x-axis

$$y=\frac{1}{x}\xrightarrow{(1)}y=\frac{1}{\left(\frac{x}{9}\right)}=\frac{9}{x}\xrightarrow{(2)}y=\frac{9}{(x-4)}\xrightarrow{(3)}(y-2)=\frac{9}{x-4}\Rightarrow y=2+\frac{9}{x-4}$$

(1) Stretch by a factor of 9 parallel to the x-axis
(2) Translate 4 units in the positive x-direction
(3) Translate 2 units in the positive y-direction

Alternative: In the sequence (1), (3), (2) **or** (3), (1), (2)

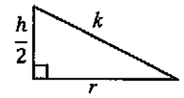
2	**Application of Differentiation (Maxima/Minima)**

2 Let V be the volume of the cylinder (in cm^3).

Steps to solve Maxima/Minima Problems

1. Draw a clear diagram and define all variables, where necessary.
2. Form equation(s) relating the variables.
3. Express quantity to be maximized/minimized in terms of a **single** variable, say x (if there are 2 variables, express one in terms of another).
4. Differentiate w.r.t. x and equate to 0 to find the stationary point(s).
5. Use 1st or 2nd derivative test to determine/prove nature of the stationary point.

By Pythagoras Theorem,



$$\left(\frac{h}{2}\right)^2+r^2=k^2$$

$$r^2=k^2-\frac{h^2}{4} \quad \text{--- (1)}$$

Question asks for **'volume of the cylinder is maximum'**
Hence the formula you need is the volume of cylinder

$$V=\pi r^2h$$
$$=\pi\left(k^2-\frac{h^2}{4}\right)h \quad \text{,from (1)}$$
$$=\pi k^2h-\frac{\pi}{4}h^3$$

Question asks for **'exact value of h'**
You should **keep h** and **get rid of r** so that you can directly solve and answer the question
Note: k is a constant (radius of sphere)

Differentiate w.r.t h,
$$\frac{dV}{dh}=\pi k^2-\frac{3\pi}{4}h^2$$

For maximum V, $\dfrac{dV}{dh}=0$

$$\therefore \frac{dV}{dh}=\pi k^2-\frac{3\pi}{4}h^2=0$$
$$k^2-\frac{3}{4}h^2=0$$
$$h^2=\frac{4k^2}{3}$$
$$h=\frac{2k}{\sqrt{3}} \quad (\because h>0)$$

Reminder: When you square root both sides of the equation, you will have ±, i.e.
$h^2=\dfrac{4k^2}{3}\Rightarrow h=\pm\dfrac{2k}{\sqrt{3}}$
So you should state why you only want $h=\dfrac{2k}{\sqrt{3}}$ from context of the question

Remember to **verify using the 1st or 2nd derivative test** that the value of **h gives the maximum volume**

Testing V is maximum

Method 1: 1st derivative test

For 1st derivative test, you need to state clearly the value of h

h	$\left(\frac{2k}{\sqrt{3}}\right)^-$	$\frac{2k}{\sqrt{3}}$	$\left(\frac{2k}{\sqrt{3}}\right)^+$
$\frac{dV}{dh}$	+	0	–
Slope	/	—	\

For 1st derivative test, you need to state the signs of $\dfrac{dV}{dh}$ and corresponding slope

$\therefore V$ is maximum when $h=\dfrac{2k}{\sqrt{3}}$.

Method 2: 2^{nd} derivative test

Differentiate w.r.t h,

$\frac{d^2V}{dh^2} = -\frac{3\pi}{2}h$

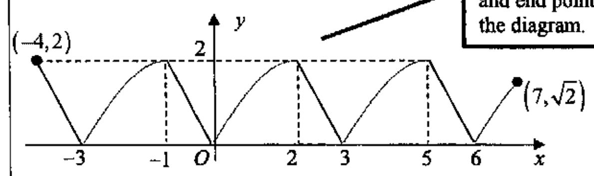
When $h = \frac{2k}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -\frac{3\pi}{2}\left(\frac{2k}{\sqrt{3}}\right) = -\sqrt{3}\pi k < 0 \quad (\because k > 0)$

$\therefore V$ is maximum when $h = \frac{2k}{\sqrt{3}}$.

For 2^{nd} derivative test, you need to
(1) find the second derivative, $\frac{d^2V}{dh^2}$
(2) explain clearly why it is < 0 for the value of h
(3) conclude that the volume is maximum

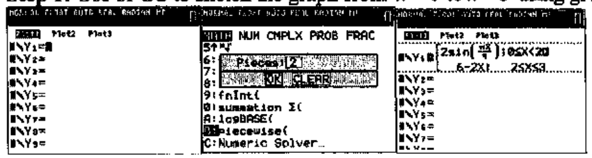
3	**Techniques of Integration**
(i)	Using $f(x) = f(x+3)$, we have $f(7) = f(4) = f(1) = 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$ or 1.41 (3s.f.)

(ii)

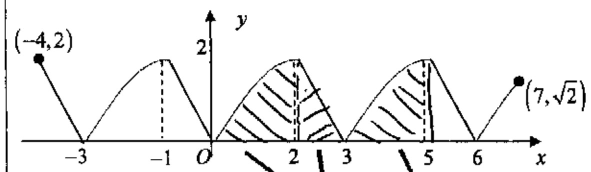


Label the axial intercepts and end points clearly on the diagram.

Step 1: Use of GC to sketch the graph from $x=0$ to $x=3$ using given equation.

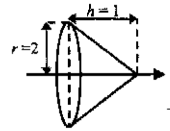


Step 2: Since $f(x) = f(x+3)$ for all real values of x, the curve is a periodic curve with a period of 3, that is, the graph repeats itself every 3 units. Thus, sketch the curve from $x = -4$ to $x = 7$ using this property.



$$\text{Volume} = \pi\int_0^5 y^2\,dx$$
$$= 2\pi\int_0^2 4\sin^2\left(\frac{\pi x}{4}\right)dx + \frac{1}{3}\pi(2)^2(1)$$
$$= 4\pi\int_0^2 1-\cos\left(\frac{\pi x}{2}\right)dx + \frac{4}{3}\pi$$
$$= 4\pi\left[x - \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^2 + \frac{4}{3}\pi$$
$$= 4\pi\left[(2-0)-0\right] + \frac{4}{3}\pi$$
$$= \frac{28}{3}\pi$$

Rotating the triangle about the x-axis gives a cone. Thus volume is found using $\frac{1}{3}\pi r^2 h$.



Note:
The equation of the curve from $x=0$ to $x=2$ is given as $y = 2\sin\left(\frac{\pi x}{4}\right)$.

The equation of the curve from $x=3$ to $x=5$ is **NOT** $y = 2\sin\left(\frac{\pi x}{4}\right)$.

Observe that the volume generated between $x=3$ to $x=5$ is identical as the volume generated between $x=0$ to $x=2$.

Thus, the volume generated between $x=0$ to $x=2$ and $x=3$ to $x=5$ is given by $2\pi\int_0^2 4\sin^2\left(\frac{\pi x}{4}\right)dx$.

Qn	Solution
4	**Maclaurin Series**

(i) Using cosine rule,

$$2^2 = r^2 + r^2 - 2r^2\cos\left(\frac{\pi}{3}-\theta\right)$$

$$4 = 2r^2 - 2r^2\cos\left(\frac{\pi}{3}-\theta\right)$$

$$r^2 = \frac{4}{2\left(1-\cos\left(\frac{\pi}{3}-\theta\right)\right)}$$

$$= \frac{4}{2\left(1-\left(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta\right)\right)}$$

$$= \frac{4}{2\left(1-\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)\right)}$$

$$= \frac{4}{2-\sqrt{3}\sin\theta-\cos\theta} \quad \text{(shown)}$$

> **Formula NOT given in MF26:**
> 1) Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos A$
> 2) Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

> Use MF26:
> $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(ii)

$$r^2 = \frac{4}{2-\sqrt{3}\sin\theta-\cos\theta}$$

$$\approx \frac{4}{2-\sqrt{3}\theta-\left(1-\frac{1}{2}\theta^2\right)}$$

$$= \frac{4}{1-\sqrt{3}\theta+\frac{1}{2}\theta^2}$$

> "θ is a sufficiently small angle" means use small angle approximation
> (θ^3 and above can be neglected)
> Use MF26:
> $\sin\theta \approx \theta$
> $\cos\theta \approx 1-\frac{\theta^2}{2}$

$$r \approx 2\left(1-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)^{-\frac{1}{2}} = 2\left(1+\left(-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)\right)^{-\frac{1}{2}}$$

> Use MF26:
> $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$

$$= 2\left(1+\left(-\frac{1}{2}\right)\left(-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)^2+\ldots\right)$$

$$= 2\left(1+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2+\frac{9}{8}\theta^2+\ldots\right)$$

$$= 2\left(1+\frac{\sqrt{3}}{2}\theta+\frac{7}{8}\theta^2+\ldots\right)$$

$$\approx 2+\sqrt{3}\theta+\frac{7}{4}\theta^2$$

$a = \sqrt{3}$ and $b = \frac{7}{4}$

> Answer question

Qn	Solution
5	**AP and GP**

(i) **Method 1**

$br = a + 10d$ ----- (1)
$br^2 = a + 2d$ ----- (2)
$br^3 = a$ ------ (3)

$$r = \frac{a}{a+2d} = \frac{a+2d}{a+10d}$$

> $r = \frac{\text{fourth term of } G}{\text{third term of } G} = \frac{a}{a+2d}$
> $r = \frac{\text{third term of } G}{\text{second term of } G} = \frac{a+2d}{a+10d}$

$a(a+10d) = (a+2d)^2$
$a^2 + 10ad = a^2 + 4ad + 4d^2$
$6ad = 4d^2$
$a = \frac{2}{3}d$ since $d \neq 0$

Hence, $r = \frac{a}{a+2d} = \frac{\frac{2}{3}d}{\frac{2}{3}d+2d} = \frac{1}{4}$

Since $|r| = \frac{1}{4} < 1$, hence the series is convergent.

> To prove geometric series G is convergent, show $|r| < 1$.

Method 2

$br = a + 10d$ ----- (1)
$br^2 = a + 2d$ ----- (2)
$br^3 = a$ ------ (3)

Eqn (2) – (1) $br^2 - br = -8d$ -----(4)
Eqn (3) – (2) $br^3 - br^2 = -2d$ -----(5)

$\frac{(5)}{(4)}$ $\frac{br^3-br^2}{br^2-br} = \frac{-2d}{-8d}$

$\frac{r^3-r^2}{r^2-r} = \frac{1}{4}$

$\frac{r^2(1-r)}{r(1-r)} = \frac{1}{4}$

Since $d \neq 0 \Rightarrow r \neq 1$, $r = \frac{1}{4}$

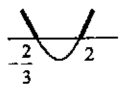
> Need to reject $r = 1$.

Since $|r| = \frac{1}{4} < 1$, hence the series is convergent.

(ii)	$H: b^2, b^2r^2, b^2r^4, \ldots$ Sum to infinity of $H = \dfrac{b^2}{1-r^2} = \dfrac{b^2}{1-\frac{1}{9}} = \dfrac{9}{8}b^2$ Sum to infinity of $G = \dfrac{b}{1-r} = \dfrac{b}{1-\frac{1}{3}} = \dfrac{3}{2}b$ $\frac{9}{8}b^2 - \frac{3}{2}b > \frac{3}{2}$ $3b^2 - 4b - 4 > 0$ $(3b+2)(b-2) > 0$ $b < -\frac{2}{3}$ or $b > 2$

> First term of $H = b^2$
> Common ratio of $H = r^2$
> Use the formula:
> $S_\infty = \dfrac{\text{first term}}{1-\text{common ratio}}$

> Sketch the graph to solve the quadratic inequality



> **Important**: Use 'or' (DO NOT use 'and')

6	**APGP + Seq & Series**
(i)	$f(r) - f(r+1) = \dfrac{1}{2^r+1} - \dfrac{1}{2^{r+1}+1}$ $= \dfrac{2^{r+1}+1-(2^r+1)}{(2^r+1)(2^{r+1}+1)}$ $= \dfrac{2(2^r)+1-2^r-1}{(2^r+1)(2^{r+1}+1)}$ $= \dfrac{2^r}{(2^r+1)(2^{r+1}+1)}$
(ii)	$\sum_{r=1}^{n} \dfrac{2^r}{(2^r+1)(2^{r+1}+1)}$ $= \sum_{r=1}^{n} [f(r) - f(r+1)]$ $= \left[\begin{array}{l} f(1) - f(2) \\ +f(2) - f(3) \\ +f(3) - f(4) \\ +\ldots \\ +f(n-1) - f(n) \\ +f(n) - f(n+1) \end{array}\right]$ $= f(1) - f(n+1)$ $= \dfrac{1}{3} - \dfrac{1}{2^{n+1}+1}$
(iii)	$\sum_{r=1}^{n} \left[\dfrac{2^{r+3}}{(2^{r+2}+1)(2^{r+3}+1)} - (2r+5)\right]$ Replace r with $r-2$ $= \sum_{r=3}^{n+2} \dfrac{2^{r+1}}{(2^r+1)(2^{r+1}+1)} - \sum_{r=1}^{n}(2r+5)$ $= 2\sum_{r=1}^{n+2} \dfrac{2^r}{(2^r+1)(2^{r+1}+1)} - 2\sum_{r=1}^{2} \dfrac{2^r}{(2^r+1)(2^{r+1}+1)} - \dfrac{n}{2}(7+2n+5)$ $= 2\left[\dfrac{1}{3} - \dfrac{1}{2^{n+3}+1}\right] - 2\left[\dfrac{1}{3} - \dfrac{1}{2^3+1}\right] - \dfrac{n(2n+12)}{2}$ $= \dfrac{2}{9} - \dfrac{2}{2^{n+3}+1} - n(n+6)$

> Need to indicate Step 3 leading to show the result stated in question.

> This part of the question was quite well done as most students were able to write out the MOD steps.

> Misconception of replacing $f(r) = \dfrac{2^r}{(2^r+1)(2^{r+1}+1)}$ instead of $\dfrac{1}{2^r+1}$.

> **Note:**
> - Need to replace the r by $(r-2)$ in the expression in (iii) instead of (ii).
> - Not recommended to replace r in (iii) by $(r+3)$ or r in (ii) by $(r+2)$. Steps will be tedious.
> - Observed and factorised the number '2' to obtain the same expression in (ii).

> Note:
> - Need to split the summation in order to **use the result in (ii)** i.e. $\sum_{r=1}^{n} \dfrac{2^r}{(2^r+1)(2^{r+1}+1)} = \dfrac{1}{3} - \dfrac{1}{2^{n+1}+1}$.
> - Do not apply MOD again as it is a "Hence" question.
> - Some applied sum of AP formula wrongly with incorrect first term or missed out $\frac{n}{2}$ or wrote $\frac{n}{2}$ (1st term – last term).

BP~736

Qn	Solution
7	**Techniques of Differentiation**
(i)	$y^2 = 3x^3 - x + 1$ Differentiate w.r.t. x, $2y\frac{dy}{dx} = 9x^2 - 1$ ← Use implicit differentiation At stationary point, $\frac{dy}{dx} = 0$ $9x^2 - 1 = 0$ $x = \frac{1}{3}$ or $-\frac{1}{3}$ When $x = \frac{1}{3}$, $y = \frac{\sqrt{7}}{3}$ or $-\frac{\sqrt{7}}{3}$ ← Always give your answer in the simplest form! i.e $\sqrt{9} = 3$ e.g. $y = \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$ or $-\sqrt{\frac{7}{9}} = -\frac{\sqrt{7}}{3}$ When $x = -\frac{1}{3}$, $y = \frac{\sqrt{11}}{3}$ or $-\frac{\sqrt{11}}{3}$ Stationary points are $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right), \left(-\frac{1}{3}, \frac{\sqrt{11}}{3}\right), \left(-\frac{1}{3}, -\frac{\sqrt{11}}{3}\right), \left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$, ← Answer the question. Give the **COORDINATES** of the stationary points. You do NOT need to determine the nature of the stationary points.
(ii)	The stationary point in the first quadrant is $S\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$. Since the x-coordinate is increasing, $\frac{dx}{dt} = \frac{1}{9}$ $\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{d\left(\frac{dy}{dx}\right)}{dx} \times \frac{dx}{dt} \Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{d^2y}{dx^2} \times \frac{dx}{dt}$ ← Important step: Use chain rule and note that $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$ $2y\frac{dy}{dx} = 9x^2 - 1$ Differentiate w.r.t. x, $2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 18x$ ← Use implicit differentiation to find the 2nd derivative $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 9x$

	At point $S\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$, $\frac{dy}{dx} = 0$, $\frac{dx}{dt} = \frac{1}{9}$ $\frac{\sqrt{7}}{3}\frac{d^2y}{dx^2} = 9\left(\frac{1}{3}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{9}{\sqrt{7}}$ $\therefore \frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{9}{\sqrt{7}} \times \left(\frac{1}{9}\right) = \frac{1}{\sqrt{7}}$ Hence the rate of change of its gradient at point S is $\frac{1}{\sqrt{7}}$ units per second.

Qn	Solution
8	**Vectors**

(i)

$\overrightarrow{OE}=\overrightarrow{OA}+\overrightarrow{AE}=\begin{pmatrix}4\\3\\0\end{pmatrix}+\begin{pmatrix}0\\0\\20\end{pmatrix}=\begin{pmatrix}4\\3\\20\end{pmatrix}$

Using Ratio Theorem,

$\overrightarrow{OF}=\dfrac{k\overrightarrow{OA}+(1-k)\overrightarrow{OE}}{k+(1-k)}=k\begin{pmatrix}4\\3\\0\end{pmatrix}+(1-k)\begin{pmatrix}4\\3\\20\end{pmatrix}=\begin{pmatrix}4\\3\\20(1-k)\end{pmatrix}$

> Since it is given that $AF:FE=1-k:k$, we can apply Ratio Theorem to find $\overrightarrow{OF}$.

(ii)

$\overrightarrow{DC}=\overrightarrow{OB}=\begin{pmatrix}0\\6\\0\end{pmatrix}=6\begin{pmatrix}0\\1\\0\end{pmatrix}$

> Find the two vectors parallel to the plane. Can use $\overrightarrow{DC}$, $\overrightarrow{DF}$ or $\overrightarrow{CF}$

$\overrightarrow{DF}=\overrightarrow{OF}-\overrightarrow{OD}$

$=\begin{pmatrix}4\\3\\20(1-k)\end{pmatrix}-\begin{pmatrix}0\\0\\20\end{pmatrix}$

$=\begin{pmatrix}4\\3\\-20k\end{pmatrix}$

$n=\overrightarrow{DC}\times\overrightarrow{DF}=\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}4\\3\\-20k\end{pmatrix}=\begin{pmatrix}-20k\\0\\-4\end{pmatrix}=-4\begin{pmatrix}5k\\0\\1\end{pmatrix}$

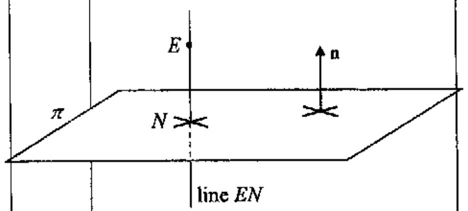
> Find the cross product of the two vectors parallel to the plane to find the normal vector

Equation of the plane *CDF*

$\mathbf{r}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\20\end{pmatrix}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20$ (Shown)

> Use $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$ to find and show the equation of the plane.
> Any one of the points *C*, *D*, or *F* can be used.

(iii) Let *N* be the foot of perpendicular from point *E* to the plane *CDF*.



> Form the equation of the line passing through points *E* and *N*. Since $\overrightarrow{EN}$ is perpendicular to the plane, we can use the normal vector of the plane to be the direction vector of the line.
>
> $l_{EN}:\mathbf{r}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\lambda\begin{pmatrix}5k\\0\\1\end{pmatrix},\lambda\in\mathbb{R}$
>
> Since *N* lies on the line, $\overrightarrow{ON}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\lambda\begin{pmatrix}5k\\0\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$
>
> Remark: Do not label the foot of perpendicular as point *F* since it is a point given in the question.

$\overrightarrow{ON}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\lambda\begin{pmatrix}5k\\0\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

$\begin{pmatrix}4+5k\lambda\\3\\20+\lambda\end{pmatrix}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20$

$5k(4+5k\lambda)+20+\lambda=20$

$20k+25k^2\lambda+\lambda=0$

$(25k^2+1)\lambda=-20k$

$\lambda=\dfrac{-20k}{(25k^2+1)}$

> Since *N* lies on the plane, it satisfies the equation of the plane. Hence $\overrightarrow{ON}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20$. Students should solve for the value of λ to find point *N*, not *k*.

The *x*-coordinate of *N* is $4+\lambda(5k)=4+\dfrac{-20k}{(25k^2+1)}(5k)=4-\dfrac{100k^2}{(25k^2+1)}=\dfrac{4}{(25k^2+1)}$

> The question only requires the *x*-coordinate of the foot of perpendicular.

Alternatively,

$\overrightarrow{ON}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\left[\dfrac{-20k}{(25k^2+1)}\right]\begin{pmatrix}5k\\0\\1\end{pmatrix}=\begin{pmatrix}4-\dfrac{100k^2}{(25k^2+1)}\\3\\20-\dfrac{20k}{(25k^2+1)}\end{pmatrix}$

The *x*-coordinate of $N=4-\dfrac{100k^2}{(25k^2+1)}=\dfrac{4}{(25k^2+1)}$

Alternative Method:

Let N be the foot of perpendicular from point E to the plane CDF.

$$\overrightarrow{ED}=\begin{pmatrix}-4\\-3\\0\end{pmatrix}$$

$$\overrightarrow{EN}=\left(\overrightarrow{ED}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$$

$$=\frac{\left[\begin{pmatrix}-4\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}\right]}{25k^2+1}\begin{pmatrix}5k\\0\\1\end{pmatrix}$$

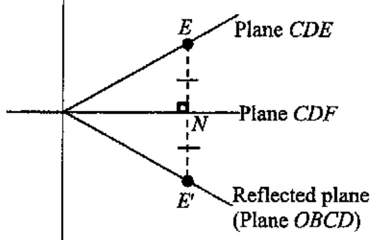
$$=\frac{-20k}{25k^2+1}\begin{pmatrix}5k\\0\\1\end{pmatrix}$$

Students using the projection method needs to be careful when applying the projection formula. It must start or end at the same point.

$$\overrightarrow{ON}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\frac{-20k}{25k^2+1}\begin{pmatrix}5k\\0\\1\end{pmatrix}=\begin{pmatrix}4-\frac{100k^2}{(25k^2+1)}\\3\\20-\frac{20k}{(25k^2+1)}\end{pmatrix}$$

The x-coordinate of $N = 4-\frac{100k^2}{(25k^2+1)}=\frac{4}{(25k^2+1)}$

(iv) Let E' be the reflection of point E in the plane CDF.



Thought process:
Point E lies on plane CDE.
We can use the foot of perpendicular from point E to plane CDF found in (iii) to find the reflected point.
Since the reflected plane lies on the plane $OBCD$, the reflected point will lie on plane $OBCD$.
From the diagram, we observe that plane $OBCD$ is the zy-plane, hence the equation of plane $OBCD$ is $x=0$ (or $\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=0$). Hence we can focus only on the x-coordinate of the reflected point and equate it to 0.

By ratio theorem,

$$\overrightarrow{ON}=\frac{\overrightarrow{OE}+\overrightarrow{OE}'}{2}$$

$$\frac{4}{(25k^2+1)}=\frac{1}{2}(4+x)$$

$$x=\frac{8}{(25k^2+1)}-4$$

The foot of perpendicular found in (iii) is the midpoint of E and E'. Use midpoint theorem to find $\overrightarrow{ON}$.
We can focus on the x-coordinate.

Alternatively,

$$\overrightarrow{OE}'=2\overrightarrow{ON}-\overrightarrow{OE}=2\begin{pmatrix}\frac{4}{(25k^2+1)}\\3\\20-\frac{20k}{(25k^2+1)}\end{pmatrix}-\begin{pmatrix}4\\3\\20\end{pmatrix}=\begin{pmatrix}\frac{8}{(25k^2+1)}-4\\3\\20-\frac{40k}{(25k^2+1)}\end{pmatrix}$$

x-coordinate of $E'=\frac{8}{(25k^2+1)}-4$

For the reflection of the plane CDE in the plane CDF to lie on the plane $OBCD$, E' has to be a point on the plane $OBCD$.

Therefore, x-coordinate of $E'=0$

$$\frac{8}{(25k^2+1)}-4=0$$

$$\frac{8}{(25k^2+1)}=4$$

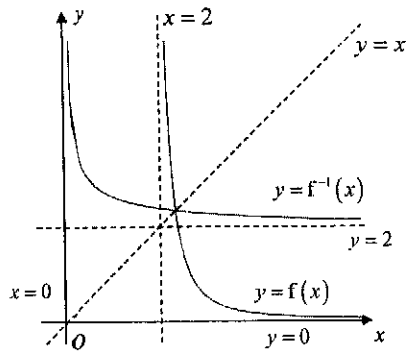
$$8=100k^2+4$$

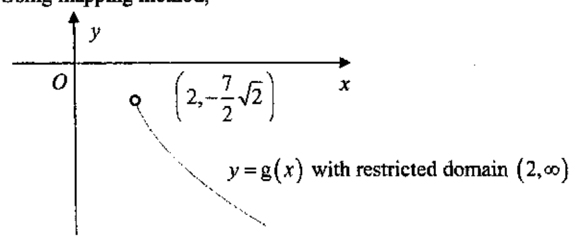
$$100k^2=4$$

$$k^2=\frac{1}{25}$$

$$k=\pm\frac{1}{5}$$

Since $0<k<1$, $k=\frac{1}{5}$

	Solution
9	**Functions + Inequalities**
(i)	For f^{-1} to exist, f must be one-one function. Least value of $k = 2$.
(ii)	 The graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$ must intersect at the same point. The asymptotes $y=2$ and $x=2$ must intersect at the line $y=x$. The asymptotes $y=0$ and $x=0$ must intersect at the line $y=x$. The graphs of $y=f(x)$ and $y=f^{-1}(x)$ must be a reflection of each other in the line $y=x$. $y=f(x)=\frac{1}{(x-2)^2}$ has asymptotes $y=0$ and $x=2$, so $y=f^{-1}(x)$ will have asymptotes $x=0$ and $y=2$.
(iii)	Let $f^{-1}(5)=x$. $ff^{-1}(5)=f(x)$ $f(x)=5$ $\frac{1}{(x-2)^2}=5$ $(x-2)^2=\frac{1}{5}$ $x-2=\frac{1}{\sqrt{5}}$ $(\because x>2)$ $x=2+\frac{1}{\sqrt{5}}$ $=2+\frac{\sqrt{5}}{5}$ First, we can compose the function f on both sides of the equation $f^{-1}(5)=x$. And since $ff^{-1}(5)=5$, we do not need to find the expression of f^{-1}.

	Alternatively, you may find the expression of f^{-1}. Let $y=\frac{1}{(x-2)^2}$ $(x-2)^2=\frac{1}{y}$ $x-2=\pm\frac{1}{\sqrt{y}}$ since $x>2$, $x=2+\frac{1}{\sqrt{y}}$ $\therefore f^{-1}(x)=2+\frac{1}{\sqrt{x}}$ $f^{-1}(5)=2+\frac{1}{\sqrt{5}}$ $=2+\frac{\sqrt{5}}{5}$
(iv)	$R_{f^{-1}}=D_f=(2,\infty)$ and $D_g=(0,\infty)$ Since $R_{f^{-1}}\subseteq D_g$, gf^{-1} exists. Using mapping method,  $D_{f^{-1}}=R_f=(0,\infty)\xrightarrow{f^{-1}}R_{f^{-1}}=(2,\infty)\xrightarrow{g}R_{gf^{-1}}=\left(-\infty,-\frac{7}{2}\sqrt{2}\right)$ Using $R_{f^{-1}}=(2,\infty)$ as the restricted domain on the graph of g, we subst when $x=2$ into $g(x)=\frac{1}{\sqrt{x}}-4\sqrt{x}$, we will get $g(2)=\frac{1}{\sqrt{2}}-4\sqrt{2}=\frac{\sqrt{2}}{2}-4\sqrt{2}=\sqrt{2}\left(\frac{1}{2}-4\right)=-\frac{7\sqrt{2}}{2}$ $R_{gf^{-1}}=\left(-\infty,-\frac{7}{2}\sqrt{2}\right)$

BP~740

(v)

$$gf(x) \le 0$$

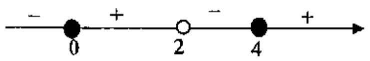
$$g\left(\frac{1}{(x-2)^2}\right) \le 0$$

$$(x-2) - \frac{4}{(x-2)} \le 0 \quad (\because x > 2)$$

$$\frac{(x-2)^2 - 4}{(x-2)} \le 0$$

$$\frac{x(x-4)}{(x-2)} \le 0$$

$$g\left(\frac{1}{(x-2)^2}\right) = \frac{1}{\sqrt{\frac{1}{(x-2)^2}}} - 4\sqrt{\frac{1}{(x-2)^2}} = \sqrt{(x-2)^2} - \frac{4}{\sqrt{(x-2)^2}} = x - 2 - \frac{4}{x-2} \quad (\text{since } x > 2)$$



$x \le 0$ or $2 < x \le 4$

Since $x > 2$, we have $2 < x \le 4$.

Since $D_{gf} = D_f$, the final range of x must satisfy the domain of f (i.e. $x > 2$)

10 Differential Equations

(i)

$$\frac{dV}{dt} = 50 - kV \quad , k > 0$$

(ii)

$$\int \frac{1}{50 - kV} dV = \int 1 \, dt$$

$$-\frac{1}{k}\ln|50 - kV| = t + c$$

$$\ln|50 - kV| = -kt - kc$$

$$50 - kV = Ae^{-kt} \quad , \text{where } A = \pm e^{-kc}$$

$$kV = 50 - Ae^{-kt}$$

$$V = \frac{50}{k} - \frac{A}{k}e^{-kt} = \frac{50}{k} - Be^{-kt} \quad , B = \frac{A}{k}$$

Use the variables separable method to ensure LHS is in terms of V, and the RHS is in terms of t

Remember the modulus sign, and the golden rule when integrating

(iii)

When $t = 0$, $V = 100 \Rightarrow 100 = \frac{50}{k} - B$ ---(1)

When $t = 1$, $V = 130 \Rightarrow 130 = \frac{50}{k} - Be^{-k}$ ---(2)

(2) – (1): $30 = -Be^{-k} + B$

$$B = \frac{30}{1 - e^{-k}}$$

$$\therefore 100 = \frac{50}{k} - \frac{30}{1 - e^{-k}}$$

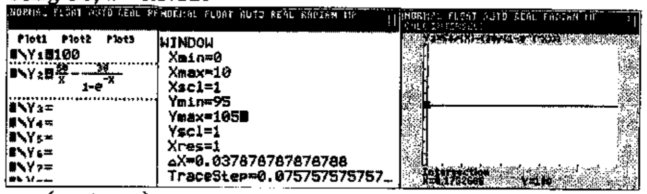
Initially, the volume of water in the tank is 100 cm^3

➔ When $t = 0$, $V = 100$

After a minute, the volume increased to 130 cm^3

➔ When $t = 1$, $V = 130$

Using GC, $k = 0.17326$



$$B = \left(\frac{30}{1 - e^{-0.17326}}\right) = 188.58$$

$$V = \frac{50}{0.17326} - 188.58e^{-0.17326t}$$

$$= 288.58 - 188.58e^{-0.17326t}$$

$$= 289 - 189e^{-0.173t} \quad (3 \text{ s.f.}) \quad (\text{shown})$$

This is a SHOW question, please show all working clearly. Intermediate answers should be given in at least 5 s.f.

(iv)

At 95% capacity, $V = 0.95(300) = 285$

$285 = 289 - 189e^{-0.173t}$

Using GC, $t = 22.3$ (3 s.f.)

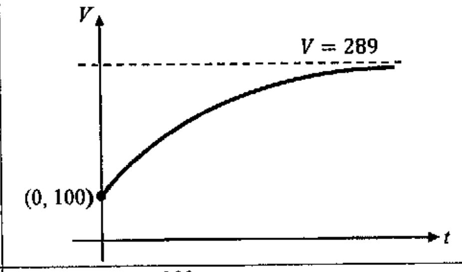
It will take about 22.3 minutes (or 22 minutes and 18 seconds) to reach 95% of the capacity.

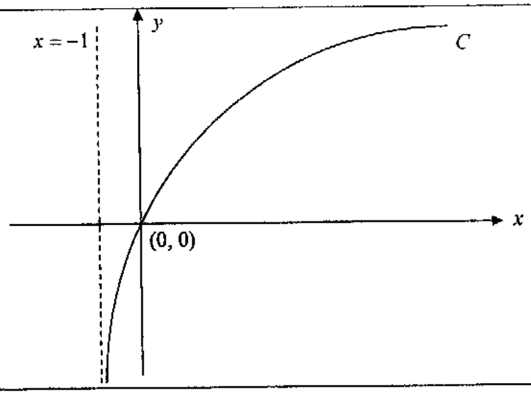
Alternatively,

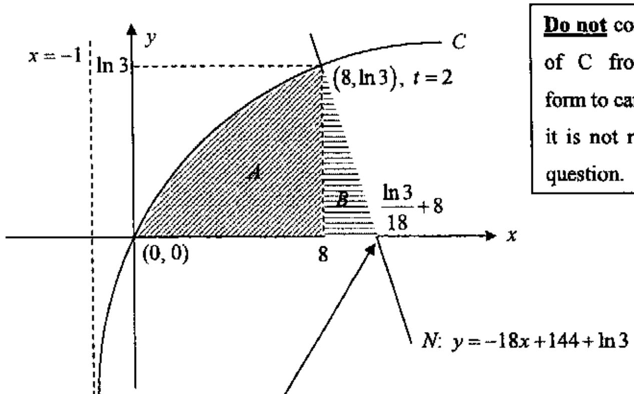
$285 = 288.58 - 188.58e^{-0.17326t}$

Using GC, $t = 22.9$ (3 s.f.)

It will take about 22.9 minutes (or 22 minutes and 54 seconds) to reach 95% of the capacity.

(v)	
(vi)	As $t \to \infty$, $V \to 289$ In the long run, the volume of the water in the tank **will increase** and **approach** 289 cm^3.

Qn	Solutions
11	
(i)	 To get vertical asymptote → observe the parametric equation of C. Since $y=\ln(t+1)$ $\Rightarrow$ vertical asymptote happens when $t+1=0 \Rightarrow t=-1$. Hence, when $t=-1$, $x=(-1)^2+2(-1) \Rightarrow x=-1$ is a vertical asymptote of C.
(ii)	$\frac{dx}{dt}=2t+2$ $\frac{dy}{dt}=\frac{1}{t+1}$ $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{1}{t+1}}{2t+2}$ $=\frac{1}{(2t+2)(t+1)}$ $=\frac{1}{2(t+1)^2}$ [$\frac{\frac{1}{t+1}}{2t+2}=\frac{1}{t+1}\div(2t+2)=\frac{1}{t+1}\times\frac{1}{2t+2}$] [**Do not** convert equation of C from parametric form to cartesian form as it is not required in the question.] When $t=2$,

	$\frac{dy}{dx}=\frac{1}{2(3)^2}=\frac{1}{18}$, $y=\ln 3$, $x=8$ Gradient of normal $=-18$ Equation of normal at $t=2$ is: $y-\ln 3=-18(x-8)$ $y=-18x+\ln 3+144$
(iii)	$\tan^{-1}18=86.8°>80°$, therefore N is not suitable in the design of the roller coaster drop as it is unsafe.
(iv)	 [**Do not** convert equation of C from parametric form to cartesian form as it is not required in the question.] When $y=0$, $0=-18x+\ln 3+144$ $x=\frac{\ln 3+144}{18}=\frac{1}{18}\ln 3+8$ [(Refer to graph) Base $=\frac{\ln 3}{18}+8-8$; Height $=\ln 3$] **Method 1:** Using x-axis Area required = Area A + Area B (Triangle) $=\int_0^8 y\,dx+\frac{1}{2}\left(\frac{1}{18}\ln 3+8-8\right)(\ln 3)$ [(Area A – change to t) 1) Limits: $x=0 \Rightarrow t=0$ | $x=8 \Rightarrow t=2$; 2) Expression: $y=\ln(t+1)$; 3) dx : $\frac{dx}{dt}=2t+2 \Rightarrow$ Let $dx=(2t+2)dt$] $=\int_0^2 \ln(t+1)(2t+2)\,dt+\frac{(\ln 3)^2}{36}$ $=\left[\left(\frac{2t^2}{2}+2t\right)\ln(t+1)\right]_0^2-\int_0^2\left(\frac{2t^2}{2}+2t\right)\frac{1}{t+1}dt+\frac{(\ln 3)^2}{36}$ $=\left[(t^2+2t)\ln(t+1)\right]_0^2-\int_0^2\frac{t^2+2t}{t+1}dt+\frac{(\ln 3)^2}{36}$ $=8\ln 3-0-\int_0^2 t+1-\frac{1}{t+1}dt+\frac{(\ln 3)^2}{36}$ [Integration by parts (LIATE) Choose $\ln(t+1)$ [L] for u; $u=\ln(t+1)$, $v=2t+2$; $\frac{du}{dt}=\frac{1}{t+1}$, $\int v\,dx=\frac{2t^2}{2}+2t$] [Long division]

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$$= 8\ln 3 - \left[\frac{t^2}{2} + t - \ln|t+1|\right]_0^2 + \frac{(\ln 3)^2}{36}$$

$$= 8\ln 3 - \left[(4-\ln 3) - 0\right] + \frac{(\ln 3)^2}{36}$$

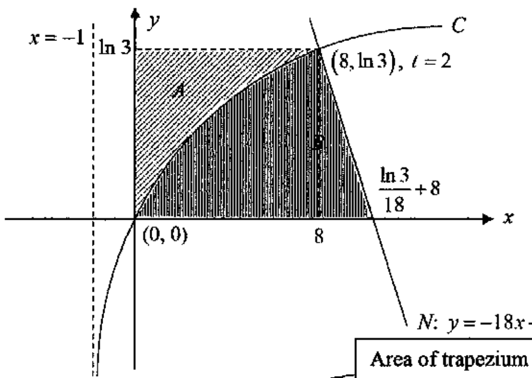
$$= \frac{1}{36}(\ln 3)^2 + 9\ln 3 - 4$$

$a = \frac{1}{36}$

$b = 9$

$c = -4$

Method 2: Using y-axis



Area of trapezium

$= \frac{1}{2} \times$ Sum of 2 parallel lines $\times$ height

$= \frac{1}{2} \times \left[\left(\frac{\ln 3}{18} + 8\right) + 8\right] \times \ln 3$

Area required
= Area $A+B$ (Trapezium) − Area A

$$= \frac{1}{2}\left(\frac{1}{18}\ln 3 + 8 + 8\right)(\ln 3) - \int_0^{\ln 3} x \, dy$$

$$= 8\ln 3 + \frac{(\ln 3)^2}{36} - \int_0^2 (t^2 + 2t)\left(\frac{1}{t+1}\right) dt$$

(Area A – change to t)

1) Limits: $y = 0 \Rightarrow t = 0$ | $y = \ln 3 \Rightarrow t = 2$

2) Expression: $x = t^2 + 2t$

3) dx : $\frac{dy}{dt} = \frac{1}{t+1} \Rightarrow$ Let $dy = \left(\frac{1}{t+1}\right) dt$

$$= 8\ln 3 + \frac{(\ln 3)^2}{36} - \int_0^2 \frac{t^2 + 2t}{t+1} dt$$

$$= 8\ln 3 + \frac{(\ln 3)^2}{36} - \int_0^2 t + 1 - \frac{1}{t+1} dt$$

Long division

$$= 8\ln 3 + \frac{(\ln 3)^2}{36} - \left[\frac{t^2}{2} + t - \ln|t+1|\right]_0^2$$

$$= 8\ln 3 + \frac{(\ln 3)^2}{36} - \left[(4-\ln 3) - 0\right]$$

$$= \frac{1}{36}(\ln 3)^2 + 9\ln 3 - 4 \quad , a = \frac{1}{36}, \; b = 9, \; c = -4$$

Qn	Solution
1	**Maclaurin's Series**
(i)	$y=\ln(1+3x+2x^2)$ $\frac{dy}{dx}=\frac{3+4x}{1+3x+2x^2}$ $(1+3x+2x^2)\frac{dy}{dx}=3+4x$ ← Useful technique: Multiply both sides by the denominator before further differentiation. **AVOID** using quotient rule for further differentiation as much as possible. diff wrt x, $(1+3x+2x^2)\frac{d^2y}{dx^2}+(3+4x)\frac{dy}{dx}=4$ (shown) ← Use implicit differentiation diff wrt x, $(1+3x+2x^2)\frac{d^3y}{dx^3}+(3+4x)\frac{d^2y}{dx^2}+4\frac{dy}{dx}+(3+4x)\frac{d^2y}{dx^2}=0$ ← Use implicit differentiation $(1+3x+2x^2)\frac{d^3y}{dx^3}+(6+8x)\frac{d^2y}{dx^2}+4\frac{dy}{dx}=0$ when $x=0$, $y=0$, $\frac{dy}{dx}=3$, $\frac{d^2y}{dx^2}=-5$, $\frac{d^3y}{dx^3}=18$, ← Find the values of f(0), f'(0), f''(0) and f ''' (0). $y=3x-\frac{5}{2!}x^2+\frac{18}{3!}x^3+...$ ← Substitute into the general form of the Maclaurin Series that can be found in MF26 $=3x-\frac{5}{2}x^2+3x^3+...$
(ii)	$y=\ln(1+3x+2x^2)$ $=3x+2x^2-\frac{(3x+2x^2)^2}{2}+\frac{(3x+2x^2)^3}{3}+...$ ← Use the standard series in MF26 for $\ln(1+x)$ up to and including the term in x^3 and show clearly that it is the same as the Maclaurin Series found in (i). $=3x+2x^2-\frac{1}{2}(9x^2+12x^3+4x^4)+\frac{1}{3}(27x^3+...)$ $=3x-\frac{5}{2}x^2+3x^3+...$ (verified) **<u>Alternative:</u>** $y=\ln(1+3x+2x^2)=\ln[(1+x)(1+2x)]$ $y=\ln(1+x)+\ln(1+2x)$ Using the standard series in MF26, $y=x-\frac{x^2}{2}+\frac{x^3}{3}+...+\left(2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}\right)$ $=3x-\frac{5}{2}x^2+3x^3+...$ (verified)

(iii)	$\ln(1+3x+2x^2)=3x-\frac{5}{2}x^2+3x^3+...$ Sub $x=\frac{1}{2}$, $\ln\left(1+3\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)^2\right)=3\left(\frac{1}{2}\right)-\frac{5}{2}\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+...$ ← Substitute $x=\frac{1}{2}$ into both sides of the Maclurin Series to get the approximate value for ln 3 $\therefore \ln 3\approx\frac{5}{4}$

Qn	Solution
2	**Vectors**
(i)	$\overrightarrow{OC} = -\frac{3}{2}\mathbf{a}$

> $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are in opposite directions, thus the negative sign is important

By ratio theorem,

$\overrightarrow{OD} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$

Area of triangle OCD

$$= \frac{1}{2}\left|\overrightarrow{OC} \times \overrightarrow{OD}\right|$$
$$= \frac{1}{2}\left|-\frac{3}{2}\mathbf{a} \times \left(\frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}\right)\right|$$
$$= \frac{3}{4}\left|\mathbf{a} \times \left(\frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}\right)\right|$$
$$= \frac{3}{4}\left|\frac{4}{5}\mathbf{a}\times\mathbf{a} + \frac{1}{5}\mathbf{a}\times\mathbf{b}\right|$$

> This is a SHOW question, there is a need to apply the distributive law and show the expansion clearly

$$= \frac{3}{4}\left|\mathbf{0} + \frac{1}{5}\mathbf{a}\times\mathbf{b}\right|$$

> Must explain that $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

$$= \frac{3}{20}\left|\mathbf{a}\times\mathbf{b}\right| \quad \text{(shown)}$$

(ii)

$$\overrightarrow{AP} = \left(\frac{\overrightarrow{AO}\cdot\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right)\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

> For the projection vector formula, there is NO modulus for $\frac{\overrightarrow{AO}\cdot\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$ as the direction is important

$$= \left(\frac{-\mathbf{a}\cdot(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}\right)\frac{(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}$$
$$= \left(\frac{-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{a}}{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})}\right)(\mathbf{b}-\mathbf{a})$$

> $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$
> $\Rightarrow |\mathbf{b}-\mathbf{a}|^2 = (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})$

$$= \left(\frac{-\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2}{|\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2}\right)(\mathbf{b}-\mathbf{a}) \quad \left(\because \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2,\ \mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2\right)$$
$$= \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2+|\mathbf{b}|^2}(\mathbf{b}-\mathbf{a}) \quad (\because \mathbf{a}\cdot\mathbf{b} = 0)$$

$$\mu = \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2+|\mathbf{b}|^2}$$

3	**Complex Numbers**
(a)(i)	**NOTE: Do not use graphing calculator when answering this question. Answers obtained using GC will not be awarded marks.**

Since coefficients of polynomial are **all** real,
$z = 2+\sqrt{3}\mathrm{i}$ is a root $\Rightarrow$ $z = 2-\sqrt{3}\mathrm{i}$ is also a root.

> State Conjugate Root Theorem properly

Quadratic factor
$$= \left(z-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(2-\sqrt{3}\mathrm{i}\right)\right)$$
$$= \left((z-2)-\sqrt{3}\mathrm{i}\right)\left((z-2)+\sqrt{3}\mathrm{i}\right)$$
$$= (z-2)^2 - (\sqrt{3}\mathrm{i})^2$$
$$= z^2 - 4z + 7$$

Let $z = b$ be the third root. (Note that the last root must be real.)
$z^3 - 8z^2 + 23z + k = \left(z^2-4z+7\right)(z-b)$

Comparing coefficient of z^2:
$-b-4 = -8 \Rightarrow b = 4$
Comparing coefficient of z^0:
$k = 7(-4) = -28$

Thus, the roots are $2+\sqrt{3}\mathrm{i}$, $2-\sqrt{3}\mathrm{i}$ and 4.

> Answer the question

Alternative Method (Not recommended)
$z^3 - 8z^2 + 23z + k = 0$
Substitute $2+\sqrt{3}\mathrm{i}$ into equation:
$$\left(2+\sqrt{3}\mathrm{i}\right)^3 - 8\left(2+\sqrt{3}\mathrm{i}\right)^2 + 23\left(2+\sqrt{3}\mathrm{i}\right) + k = 0$$
$$\left(8+12\sqrt{3}\mathrm{i}+6\left(\sqrt{3}\mathrm{i}\right)^2+\left(\sqrt{3}\mathrm{i}\right)^3\right) - 8\left(4+4\sqrt{3}\mathrm{i}-\sqrt{3}^2\right) + 46 + 23\sqrt{3}\mathrm{i} + k = 0$$
$$8+12\sqrt{3}\mathrm{i}-18-3\sqrt{3}\mathrm{i}-32-32\sqrt{3}\mathrm{i}+24+46+23\sqrt{3}\mathrm{i}+k = 0$$
$$28 + k = 0$$
$$k = -28$$

> GC is not allowed so working needs to be shown clearly

> No mark if working is not shown clearly

Since coefficients of polynomial are **all** real,
$z = 2+\sqrt{3}\mathrm{i}$ is a root $\Rightarrow$ $z = 2-\sqrt{3}\mathrm{i}$ is also a root.

> State Conjugate Root Theorem properly

Quadratic factor
$$= \left(z-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(2-\sqrt{3}\mathrm{i}\right)\right)$$
$$= \left((z-2)-\sqrt{3}\mathrm{i}\right)\left((z-2)+\sqrt{3}\mathrm{i}\right)$$
$$= (z-2)^2 - (\sqrt{3}\mathrm{i})^2$$
$$= z^2 - 4z + 7$$

Let $z = b$ be the third root. (Note that the last root must be real.)

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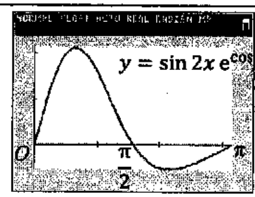
	$z^3-8z^2+23z-28=(z^2-4z+7)(z-b)$ Comparing constant $-b-4=-8 \Rightarrow b=4$ Thus, the roots are $2+\sqrt{3}i$, $2-\sqrt{3}i$ and 4. [Answer the question]
(a)(ii)	Replace z with iz $iz=2+\sqrt{3}i$ or $iz=2-\sqrt{3}i$ or $iz=4$ $z=\sqrt{3}-2i$ or $z=\sqrt{3}-2i$ or $z=-4i$ ["Hence" so use previous part's answer and do a replacement]
(b)	$w=\cos\theta+i\sin\theta=e^{i\theta}$ $\frac{w^*}{w^2+1}=\frac{e^{i(-\theta)}}{e^{i(2\theta)}+1}$ [Tip: Since this involves division of 2 complex numbers, use exponential form] $=\frac{e^{i(-\theta)}}{e^{i(\theta)}\left(e^{i\theta}+e^{i(-\theta)}\right)}$ [Useful result: $e^{i(2\theta)}=e^{i(\theta)}\left(e^{i\theta}+e^{i(-\theta)}\right)$] $=\frac{e^{i(-2\theta)}}{2\cos\theta}$ [$e^{i\theta}+e^{i(-\theta)}=\cos\theta+i\sin\theta+\cos\theta-i\sin\theta=2\cos\theta$] $=\frac{1}{2}\sec(\theta)e^{i(-2\theta)}$ [This is of the form $Re^{i(-2\theta)}$ where $R=\frac{1}{2}\sec\theta$ is the modulus and -2θ is the argument] $\left\lvert\frac{w^*}{w^2+1}\right\rvert=\frac{1}{2}\sec\theta$ $\arg\left(\frac{w^*}{w^2+1}\right)=-2\theta$

4	**Integration**
(a)(i)	$\frac{d}{dx}\left(e^{\cos x}\right)=-\sin x e^{\cos x}$
(ii)	**Note:** $\frac{d}{dx}\left(e^{\cos x}\right)=-\sin x\, e^{\cos x} \Rightarrow \int \sin x\, e^{\cos x}dx=-e^{\cos x}+C$ [For integration questions, if question asks you to differentiate something first, it is to guide you to see the integration] [Double angle formula (MF26): $\sin 2A = 2\sin A\cos A$] $\int \sin 2x\, e^{\cos x}\,dx$ $=\int 2\sin x\cos x\, e^{\cos x}\,dx$ $=2\int \cos x\left(\sin x\, e^{\cos x}\right)dx$ K I − D I [$u=\cos x$, $v=(\sin x)e^{\cos x}$, $\frac{du}{dx}=-\sin x$, $\int v\,dx=-e^{\cos x}$] $=2\left[\cos x\left(-e^{\cos x}\right)-\int(-\sin x)\left(-e^{\cos x}\right)dx\right]$ $=2\left[-\cos x\, e^{\cos x}-\int \sin x\, e^{\cos x}dx\right]$ $=2\left[-\cos x\, e^{\cos x}+e^{\cos x}\right]+C$ $=2e^{\cos x}(1-\cos x)+C$ (shown) [For integration by parts, **Highly recommended** to work out the u and v at the side, then apply KI – DI to fill in each part accordingly]
(iii)	$\int_0^{\pi}\left\lvert\sin 2x\, e^{\cos x}\right\rvert dx$ [**Keyword: exact** ⇒Cannot just use G.C. Need to remove the modulus before you can integrate] [From the graph of $y=\sin 2x\, e^{\cos x}$, observe that when $y\le 0$, $\frac{\pi}{2}\le x\le\pi$ $\therefore \left\lvert\sin 2x\, e^{\cos x}\right\rvert=\begin{cases}\sin 2x\, e^{\cos x} & ,0\le x<\frac{\pi}{2}\\ -\sin 2x\, e^{\cos x} & ,\frac{\pi}{2}\le x\le\pi\end{cases}$]  $=\int_0^{\frac{\pi}{2}}\sin 2x\, e^{\cos x}\,dx-\int_{\frac{\pi}{2}}^{\pi}\sin 2x\, e^{\cos x}\,dx$ $=\left[2e^{\cos x}(1-\cos x)\right]_0^{\frac{\pi}{2}}-\left[2e^{\cos x}(1-\cos x)\right]_{\frac{\pi}{2}}^{\pi}$ [Use the show result from **(ii)**] $=2e^0(1-0)-2e^1(1-1)-\left[2e^{-1}(1+1)-2e^0(1-0)\right]$ $=4-4e^{-1}$

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(b)

$x=\sec\theta \Rightarrow \dfrac{dx}{d\theta}=\sec\theta\tan\theta$ ← MF26

$\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx$

$\displaystyle=\int \frac{1}{\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,d\theta$

Recall: $\tan^2\theta+1=\sec^2\theta$

$\displaystyle=\int \frac{1}{\sqrt{\tan^2\theta}}(\sec\theta\tan\theta)\,d\theta$

$\displaystyle=\int \frac{1}{\tan\theta}(\sec\theta\tan\theta)\,d\theta$ since $0<\theta<\dfrac{\pi}{2}$

$\displaystyle=\int \sec\theta\,d\theta$

MF26

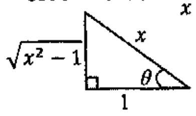
$=\ln|\sec\theta+\tan\theta|+C$

You should remove the modulus when possible

$=\ln(\sec\theta+\tan\theta)+C$ since $0<\theta<\dfrac{\pi}{2}\Rightarrow \sec\theta>0, \tan\theta>0$

$=\ln\left(x+\sqrt{x^2-1}\right)+C$ ← Since it is indefinite integral, need to change back to x and remember the '+ C'

$x=\sec\theta\Rightarrow\cos\theta=\dfrac{1}{x}$



$\therefore \tan\theta=\dfrac{\sqrt{x^2-1}}{1}$

Steps for integration by substitution:

1) Differentiate the given substitution
   - Hidden working:
     (__)dx = (__)dθ
     Everything involving x | Everything involving θ
2) Substitute everything to θ in the integration in 1 step:
   - Expression (including the dx)
   - Upper & Lower limits (if any)
3) Simplify and integrate accordingly
4) For indefinite integrals, substitute back the x accordingly

5 Probability

(i)

$P(B)=P(A\cup B)-P(A\cap B')$

$=\dfrac{9}{16}-\dfrac{1}{16}$

$=\dfrac{8}{16}=\dfrac{1}{2}$

(ii)

$P(A'\cap B)=\dfrac{1}{2}\left(1-\dfrac{1}{8}\right)=\dfrac{7}{16}$

$P(A\cap B)=\dfrac{1}{2}-\dfrac{7}{16}=\dfrac{1}{16}$ and $P(A)P(B)=\dfrac{1}{8}\left(\dfrac{1}{2}\right)=\dfrac{1}{16}$

Since $P(A)P(B)=P(A\cap B)=\dfrac{1}{16}$, the events A and B are independent.

Alternatively

Given that $P(A\cap B')=\dfrac{1}{16}$ and $P(A)\times P(B')=\dfrac{1}{8}\times\dfrac{1}{2}=\dfrac{1}{16}$

Since $P(A)P(B')=P(A\cap B')=\dfrac{1}{16}$, the <u>events A and B' are independent</u>.

$\therefore$ the events A and B are independent.

(iii)

Since C is independent of A, $\therefore$ the events of A' and C are independent.

$P(A'\cap C)=P(A')\times P(C)$

$=\left(1-\dfrac{1}{8}\right)\left(\dfrac{3}{8}\right)$

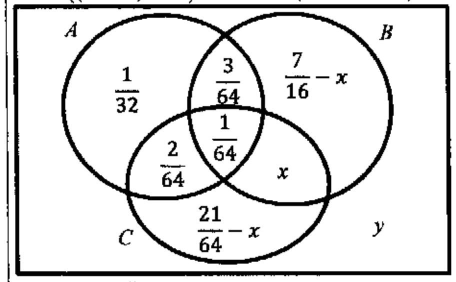
$=\dfrac{21}{64}$

(iv)

Since A and B are independent, A' and B are also independent,

$P(A'\cap B)=P(A')P(B)=\dfrac{7}{8}\left(\dfrac{1}{2}\right)=\dfrac{7}{16}$

Let $P((B\cap C)\cap A')=x$ and $P(A'\cap B'\cap C')=y$



From Venn diagram,

$\dfrac{1}{32}+\dfrac{3}{64}+\dfrac{1}{64}+\dfrac{2}{64}+\dfrac{7}{16}-x+x+\dfrac{21}{64}-x+y=1$

$\dfrac{57}{64}-x+y=1$

$y=\frac{7}{64}+x$

For minimum y, minimum $x=0$, $\Rightarrow y=\frac{7}{64}$

For maximum y, maximum $x=\frac{21}{64}$, $\Rightarrow y=\frac{7}{64}+\frac{21}{64}=\frac{7}{16}$

$\therefore \frac{7}{64}\le \mathrm{P}(A'\cap B'\cap C')\le \frac{7}{16}$

6 DRV

(a)(i)

$p+2p+5p=1$ [Total sum of probabilities = 1]

$p=\frac{1}{8}$

$\mathrm{P}(X=6)=\left(\frac{5}{8}\right)\left(\frac{1}{8}\right)2!+\left(\frac{2}{8}\right)^2=\frac{7}{32}$

Case 1: 1 Bull's eye and 1 outer ring

$\left(\frac{5}{8}\right)\left(\frac{1}{8}\right)2!$ ← No of cases = No of ways to permute 1 Bull's eye and 1 outer ring

Case 2: 2 inner ring

$\left(\frac{2}{8}\right)^2$ ← **There is only one case.**

$\mathrm{P}(X=2)=\left(\frac{5}{8}\right)^2=\frac{25}{64}$.

$\mathrm{P}(X=4)=\left(\frac{5}{8}\right)\left(\frac{2}{8}\right)2!=\frac{5}{16}$.

$\mathrm{P}(X=8)=\left(\frac{1}{8}\right)\left(\frac{2}{8}\right)2!=\frac{1}{16}$.

$\mathrm{P}(X=10)=\left(\frac{1}{8}\right)^2=\frac{1}{64}$.

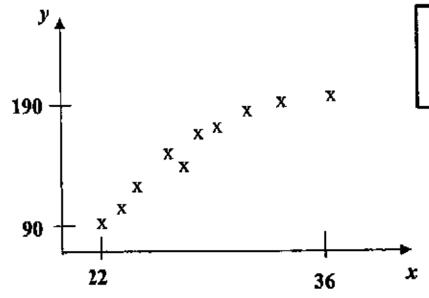
x	2	4	6	8	10
$\mathrm{P}(X=x)$	$\frac{25}{64}$	$\frac{5}{16}$	$\frac{7}{32}$	$\frac{1}{16}$	$\frac{1}{64}$

(ii) Using GC, $\mathrm{E}(X)=4$

(iii) For the game to be fair, $\mathrm{E}\left(\frac{X}{2}-k\right)=0\Rightarrow \frac{1}{2}\mathrm{E}(X)-k=0\Rightarrow k=2$.

[Expected payout] [Cost of 1 round]

Idea: Fair game refers to no loss or profit from playing the game.
Hence, expected payout (expectation) − cost of 1 round = 0

Qn	Solution
7	**Correlation and Regression**
(i)	 Only some did not label the max and min values on scatter diagram. From the scatter diagram, as the **daily average temperature increases**, the **number of popsicles sold increases at a decreasing rate**. (or as **x increases, y increases at a decreasing rate**) Note: Comment the relationship from scatter diagram instead of using r value.
(ii)	For $y = a + bx$, r value $= 0.95143 = 0.951$ (3 s.f.) For $\ln(200 - y) = c + dx$, r value $= -0.98920 = -0.989$ (3s.f.) Since $\lvert r\rvert = 0.989$ for the model $\ln(200-y) = c + dx$ is closer to 1 than the $\lvert r\rvert = 0.951$ for the model $y = a + bx$, hence $\ln(200 - y) = c + dx$ is a better model. **OR** Since the product moment correlation coefficient for the model $\ln(200-y) = c + dx$ of -0.989 is closer to -1 than the product moment correlation coefficient for the model $y = a + bx$ of 0.951 to 1, hence $\ln(200-y) = c + dx$ is a better model. $\ln(200 - y) = 8.7493 - 0.18032x = 8.75 - 0.180x$ Need to compare the $\lvert r\rvert$ value of both models or r values of both models.
(iii)	When $x = 37$, $y = 192$ (3 s.f.) The estimate is not reliable as $x = 37$ is outside the data range $22 \le x \le 36$ and a linear model between $\ln(200 - y)$ and x may not hold. Need to justify that the value of 37 is outside the data range.
(iv)	The value of 200 is the theoretical maximum number of popsicles that the store can sell per day. **Keywords: theoretical maximum number** of popsicles sold or the number of popsicles sold will **approach 200**.

Qn	Solution	Comments
8	**Hypothesis Testing**	**Comments**
(i)	Let X be the mass of a randomly chosen orange (in grams). Let μ denote the population mean mass of an orange (in grams).	Define the variables accurately.
	$s^2 = \frac{40}{40-1}(4^2) = \frac{640}{39} = 16.410$ $\bar{x} = 364.2$	4 is sample standard deviation. Use the formula $s^2 = \frac{n}{n-1} \times$ sample variance to find s^2.
	$H_0 : \mu = 365$ $H_1 : \mu < 365$	The fruit seller claims that the mean mass is 365g. To test if he has overstated the mean, we test if the mean is less than 365g.
	Under H_0, since $n = 40$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(365, \frac{\left(\frac{640}{39}\right)}{40}\right)$ approximately.	Since the distribution of X is unknown and the sample size is large, we can use Central Limit Theorem to find the distribution of $\bar{X}$. Ensure phrasing is complete and CLT is spelt out in full.
	Test statistic: $Z = \frac{\bar{X} - 365}{\sqrt{\frac{\left(640/39\right)}{40}}} \sim N(0,1)$	Standardise $\bar{X}$ to obtain the test statistic.
	Level of significance: 5% Reject H_0 if p-value < 0.05	Write down the level of significance and the rejection criteria. We can use p-value since there are no unknowns in the question.
	Under H_0, using G.C., p-value $= 0.10583$	
	Since p-value $= 0.106 > 0.05$, we do not reject H_0 and conclude that there is insufficient evidence at 5% level, that the population mean mass of the orange is less than 365g. Hence, the fruit seller has not overstated the mean at 5% level.	Take note of what is necessary in the conclusion: 1. Reject/do not reject H_0 2. Level of significance 3. Sufficient/insufficient evidence for H_1 (written in context) The question is whether the mean is overstated, not whether the fruit seller's claim is valid. Answer the question.

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(ii) Let D be the diameter of a randomly chosen orange (in cm).
Let μ denote the population mean diameter of an orange (in cm).

> Define the variables accurately. Note that D is defined in the question and does not have to be redefined.

$H_0 : \mu = 9$

$H_1 : \mu > 9$

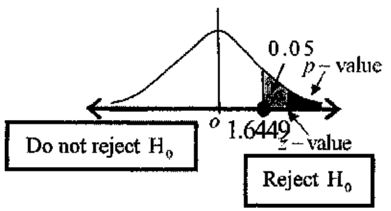
> 'Expected value of D is 9' means that 'the mean of D is 9'. In particular, we are testing if the mean diameter exceeds 9.

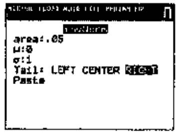
Under H_0, $D \sim N(9, 0.3^2)$, $\therefore \overline{D} \sim N\left(9, \frac{0.3^2}{50}\right)$.

Test Statistic: $Z = \frac{\overline{D}-9}{\sqrt{\frac{0.3^2}{50}}} \sim N(0,1)$

Level of significance: 5%

> Write down the reason and the distribution of $\overline{D}$ clearly. It is stated that D is normally distributed and the population standard deviation is given to be 0.3.





> Write down the rejection criteria for the right-tail test at 5% level of significance. We have to use z-value method as sample mean is unknown. Represent it using $\bar{d}$.

Reject H_0 if z-value > 1.6449

Let $\bar{d}$ denote the sample mean diameter of an orange for this sample.

z-value $= \frac{\bar{d}-9}{\left(0.3/\sqrt{50}\right)}$

Since there is sufficient evidence to reject H_0,

$\therefore$ z-value > 1.6449

$\frac{\bar{d}-9}{\left(0.3/\sqrt{50}\right)} > 1.6449$

$\bar{d} > 9.0698$

$\therefore \bar{d} > 9.07$ (3 s.f.)

> 'Sufficient evident from the sample to conclude that the mean diameter exceeds 9cm' means that there is sufficient evidence for H_1 i.e. H_0 is rejected. Solve for the unknown $\bar{d}$. Round off the answer to satisfy the inequality.

(iii) Since $\bar{d} = 8.9 < 9.07$ is outside the rejection region found in **(ii)**, we do not reject H_0 and conclude that there is insufficient evidence, at 5% level of significance, that the population mean diameter of the orange is more than 9 cm.

> Take note of what is necessary in the conclusion:
> 1. Reject/do not reject H_0
> 2. Level of significance
> 3. Sufficient/insufficient evidence for H_1

> Note that the hypotheses and sample size for **(ii)** and **(iii)** are the same. The population variance is also constant. Hence we can use the result in **(ii)**.
>
> From **(ii)**, for H_0 to be rejected, the sample mean is greater than 9.07. Since the sample mean is less than 9.07 for this part, we do not reject H_0. Ensure that clear reasoning is given and conclusion is complete.

Solution

9 Binomial Distribution & Probability

(i) Assumptions:

> For Binomial Distribution, need to explain the following assumptions **in the context of the question**
> 1. Independent trials (NOT independent probability)
> 2. Constant probability of success
>
> The other 2 conditions (fixed number of trials and two mutually exclusive outcomes) are implied to be true from the question.

1. Whether a randomly chosen ball tossed by the participant goes into the container is **independent** of any other balls tossed.

2. The **probability** that a randomly chosen ball tossed by the participant goes into the container is **constant** at 0.35.

The participant may adjust the toss after he succeeds or fails, hence the condition of independence may not hold.

> Reason given must explain clearly why an assumption may not hold by relating to a round of the game played by a participant.

(ii) $X \sim B(5, 0.35)$

$P(X \geq 3) = 1 - P(X \leq 2)$

$= 0.23517$

$= 0.235$

> Important to use complement to change to $1 - P(X \leq 2)$ so that GC Binomcdf command can be used

(iii) Let Y be the number of participants, out of 10, who did not win a prize.

$Y \sim B(10, 1-0.23517) \Rightarrow Y \sim B(10, 0.76483)$

$P(Y \leq 6) = 0.191$

> Use 5 s.f value

> **Important:** define all variables clearly

Altenative solution:

Let W be the number of participants, out of 10, who won a prize.

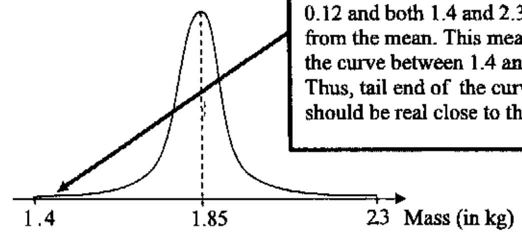
$W \sim B(10, 0.23517)$

$P(W \geq 4) = 1 - P(W \leq 3)$

$= 0.191$

> At most 6 (out of 10) did not win a prize **is equivalent to** At least 4 (out of 10) won a prize

(iv)	$P(\text{Wins a prize})$ $= P(X \ge 3) + P(X = 2)P(X \ge 4)$ $= (0.23517) + (0.33642)(0.054023)$ $= 0.25334$ $= 0.253$ Consider 2 cases for a participant to win a prize: **Case 1:** participant tosses at least 3 balls into the container in 1st round only **Case 2:** participant tosses exactly 2 balls into the container in 1st round **and** tosses at least 4 balls into the container in 2nd round.
(v)	$P(\text{Toss fewer than 7 balls into the container} \mid \text{Wins a prize})$ $= \dfrac{P(\text{Toss fewer than 7 balls into the container and wins a prize})}{P(\text{Wins a prize})}$ $= \dfrac{P(X \ge 3) + P(X = 2)P(X = 4)}{0.25334}$ $= \dfrac{0.25128}{0.25334}$ $= 0.993$ Use conditional probability For the numerator, consider 2 cases for a participant to toss fewer than 7 balls and win a prize **Case 1:** participant tosses at least 3 balls into the container in 1st round only **Case 2:** participant tosses exactly 2 balls into the container in 1st round **and** tosses **exactly 4** balls into the container in 2nd round. For the denominator, use the P(wins a prize) value (up to at least 5 s.f) found in **part (iv).**

10	**Normal and Sampling Distributions**
(i)	 Note that the standard deviation (SD) is 0.12 and both 1.4 and 2.3 is 3.75 SD away from the mean. This meant the area under the curve between 1.4 and 2.3 is close to 1. Thus, tail end of the curve at 1.4 and 2.3 should be real close to the axis.
(ii)	Let X be the mass of a randomly chosen papaya (in kg). $X \sim N(1.85, 0.12^2)$ Let Y be the mass of a randomly chosen watermelon (in kg). $Y \sim N(6.5, 0.72^2)$ $P(X < 1.7) = 0.106$ (3 s.f.) **Note:** You must: a) Define the variables used clearly and b) Write down the distribution with the parameter values evaluated.
(iii)	$E(Y - X) = 6.5 - 1.85 = 4.65$ $\text{Var}(Y - X) = 0.72^2 + 0.12^2 = 0.5328$ $Y - X \sim N(4.65, 0.5328)$ $P(Y - X > 4.5) = 0.581$ (3 s.f.) **Note:** a) $\text{Var}(Y - X) = \text{Var}(Y) + \text{Var}(X)$ b) 0.5328 is an exact answer. Do NOT round off to 3 s.f.
(iv)	$\overline{Y} = \dfrac{Y_1 + Y_2 + \cdots Y_n}{n} \sim N\left(6.5, \dfrac{0.72^2}{n}\right)$ $P(\overline{Y} > k) = 0.2$ $P\left(Z > \dfrac{k - 6.5}{0.72/\sqrt{n}}\right) = 0.2$ $\dfrac{k - 6.5}{0.72/\sqrt{n}} = 0.84162$ $k = 6.5 + \dfrac{0.606}{\sqrt{n}}$
(v)	Let $T = 2.20(X_1 + X_2 + X_3) + 1.45(Y_1 + Y_2 + Y_3 + Y_4)$ $E(T) = 2.20 \times 3 \times 1.85 + 1.45 \times 4 \times 6.5 = 49.91$ $\text{Var}(T) = 2.20^2 \times 3 \times 0.12^2 + 1.45^2 \times 4 \times 0.72^2 = 4.568832$ $T \sim N(49.91, 4.568832)$ $P(T > 50) = 0.483$ (3 s.f.) **Note**: Do NOT round off exact answer to 3 s.f. In this question, 49.91 and 4.568832 are both exact answers.
(vi)	We assume that the **distributions** of the **masses** of **all** fruits are **independent** of one another. **Note:** The key words are in bold. **Common Mistake**: Phrasings by students are not clear in illustrating the need of independence between the distribution of masses of any two randomly chosen papayas (watermelons).

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