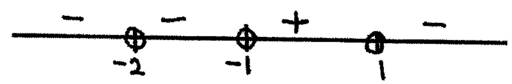
2025 H2MA Prelim Paper 1

1	Solution [4] Complex Number	
(a)	<u>Method 1</u> $f(-1)=0$ $a(-1)^3+b(-1)^2+c(-1)+d=0$ $-a+b-c+d=0$ $b-c+d-a=0$ ------------------ (1) $f(4+i)=0$ $a(4+i)^3+b(4+i)^2+c(4+i)+d=0$ $a(52+47i)+b(15+8i)+c(4+i)+d=0$ Comparing real parts, $15b+4c+d+52a=0$ ----------------- (2) Comparing imaginary parts, $8b+c+47a=0$ ------------------(3) Solving (1), (2) and (3) using the GC, $b=-7a$, $c=9a$ and $d=17a$ <u>Method 2</u> Since all the coefficients of $f(x)$ are real, $4+i$ and $4-i$ are conjugate roots of $f(x)=0$. $f(x)=ax^3+bx^2+cx+d$ $=a[x-(4+i)][x-(4-i)][x+1]$ $=a[x^2-8x+17][x+1]$ $=a[x^3-7x^2+9x+17]$ $=ax^3-7ax^2+9ax+17a$ $b=-7a$, $c=9a$ and $d=17a$	

2	Solution [6] Inequality	
	$\frac{9}{1-x^2}<\frac{x+5}{x+1}$ $\frac{9-(x+5)(1-x)}{(x+1)(1-x)}<0$ $\frac{9-(-x^2-4x+5)}{(x+1)(1-x)}<0$ $\frac{x^2+4x+4}{(x+1)(1-x)}<0$ $\frac{(x+2)^2}{(x+1)(1-x)}<0$  $x<-1$ or $x>1$, $x\neq-2$ (Alternatively, $x<-2$ or $-2<x<-1$ or $x>1$)	
	$\frac{9}{1-e^{2x}}<\frac{e^x+5}{e^x+1}$. Replace x with e^x in the previous inequality, $e^x<-1$ or $e^x>1$ (no solution since $e^x>0$ for $x\in\mathbb{R}$) $x>0$ (Note $e^x\neq-2$ is always true) Thus $x>0$. <u>Alternative presentation:</u> $e^x<-2$ or $-2<e^x<-1$ or $e^x>1$ (no solution since $e^x>0$ for $x\in\mathbb{R}$) $x>0$ Thus $x>0$.	

3 Solution [6] Complex Numbers

$z^2-(1+2i)z+1+7i=0$

In the above equation, $a=1,\ b=-1-2\text{i},\ c=1+7\text{i}$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{1+2\text{i}\pm\sqrt{(-1-2\text{i})^2-4(1)(1+7\text{i})}}{2(1)}$$

$$=\frac{1+2\text{i}\pm\sqrt{4\text{i}-3-4-28\text{i}}}{2}$$

$$=\frac{1+2\text{i}\pm\sqrt{-7-24\text{i}}}{2}$$

We want to find $\sqrt{-7-24\text{i}}$.

Let $\sqrt{-7-24\text{i}}=x+\text{i}y,\ x,y\in\mathbb{R}$

$-7-24\text{i}=(x+\text{i}y)^2$ ---(*)

$-7-24\text{i}=(x^2-y^2)+(2xy)\text{i}$

Comparing real and imaginary parts,

$x^2-y^2=-7$ —(1)

$2xy=-24$

$x=\frac{-12}{y}$ ---(2)

Substitute (2) into (1):

$\left(\frac{-12}{y}\right)^2-y^2=-7$

$\frac{144}{y^2}-y^2=-7$

$144-y^4=-7y^2$

$y^4-7y^2-144=0$

$(y^2-16)(y^2+9)=0$

$y^2-16=0$ or $y^2+9=0$

$y^2=16$ $\quad$ $y^2=-9$

$y=\pm4$ $\quad$ (reject as $y\in\mathbb{R}$)

When $y=4,\ x=-3$.

When $y=-4,\ x=3$

The square roots of $-7-24\text{i}$ are $3-4\text{i}$ and $-3+4\text{i}$.

Hence, $z=\frac{1+2\text{i}\pm(3-4\text{i})}{2}$

$z=\frac{1+2\text{i}+3-4\text{i}}{2}$ or $z=\frac{1+2\text{i}-3+4\text{i}}{2}$

$z=2-\text{i}$ or $z=-1+3\text{i}$

4 Solution [6] Summation

(a)

$$\sum_{r=1}^{N}\frac{1}{(r+1)(r+2)}$$
$$=\sum_{k-1=1}^{k-1=N}\frac{1}{((k-1)+1)((k-1)+2)}$$
$$=\sum_{k=2}^{N+1}\frac{1}{k(k+1)}$$
$$=\left(1-\frac{1}{(N+1)+1}\right)-\frac{1}{(1)(1+1)}$$
$$=1-\frac{1}{N+2}-\frac{1}{2}$$
$$=\frac{1}{2}-\frac{1}{N+2}$$

(b)

$$8\sum_{r=k+1}^{\infty}\frac{1}{r(r+1)}=\sum_{r=1}^{k}\frac{1}{r(r+1)}$$
$$8\left(\sum_{r=1}^{\infty}\frac{1}{r(r+1)}-\sum_{r=1}^{k}\frac{1}{r(r+1)}\right)=\sum_{r=1}^{k}\frac{1}{r(r+1)}$$
$$8\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=9\sum_{r=1}^{k}\frac{1}{r(r+1)}$$

As $n\to\infty$, $\frac{1}{n+1}\to 0$. Hence $\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=1$.

$$8(1)=9\sum_{r=1}^{k}\frac{1}{r(r+1)}$$
$$\sum_{r=1}^{k}\frac{1}{r(r+1)}=\frac{8}{9}$$
$$1-\frac{1}{k+1}=\frac{8}{9}$$
$$\frac{k}{k+1}=\frac{8}{9}$$
$$k=8$$

5 Solution [7] Integration by parts

(a)

$$\int\frac{x}{\sqrt{16-x^4}}\,dx=\frac{1}{2}\int\frac{2x}{\sqrt{(4)^2-(x^2)^2}}\,dx$$
$$=\frac{1}{2}\sin^{-1}\left(\frac{x^2}{4}\right)+c$$

(b)

Let $u=\frac{1}{2}\sin^{-1}\left(\frac{x^2}{4}\right)$, $\frac{dv}{dx}=x$

$\frac{du}{dx}=\frac{x}{\sqrt{16-x^4}}$ $\quad v=\frac{1}{2}x^2$

$$\int\frac{x}{2}\sin^{-1}\left(\frac{x^2}{4}\right)dx=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)-\int\frac{x^2}{2}\cdot\frac{x}{\sqrt{16-x^4}}\,dx$$
$$=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)-\frac{1}{2}\int\frac{x^3}{\sqrt{16-x^4}}\,dx$$
$$=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)-\frac{1}{2}\left(-\frac{1}{4}\right)\int\frac{-4x^3}{\sqrt{16-x^4}}\,dx$$
$$=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)-\frac{\frac{1}{2}\left(-\frac{1}{4}\right)}{\frac{1}{2}}\sqrt{16-x^4}+c$$
$$=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\frac{1}{4}\sqrt{16-x^4}+c$$

<u>Alternative 1</u>

	Let $u=\sin^{-1}\left(\frac{x^2}{4}\right)$, $\frac{dv}{dx}=\frac{x}{2}$ $\frac{du}{dx}=\frac{\frac{1}{2}x}{\sqrt{1-\frac{x^4}{16}}}$ $\quad v=\frac{x^2}{4}$ $\int\frac{x}{2}\sin^{-1}\left(\frac{x^2}{4}\right)dx$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)-\int\frac{x^2}{4}\cdot\frac{\frac{1}{2}x}{\sqrt{1-\frac{x^4}{16}}}dx$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\frac{1}{2}\int\frac{-\frac{1}{4}x^3}{\sqrt{1-\frac{x^4}{16}}}dx$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\frac{1}{2}\frac{\sqrt{1-\frac{x^4}{16}}}{\frac{1}{2}}+C$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\sqrt{1-\frac{x^4}{16}}+C$	
	Alternative #2 Consider $\int\sin^{-1}(x)\,dx$ Let $u=\sin^{-1}(x)$, $\quad\frac{dv}{dx}=1$ $\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$ $\quad v=x$	

	$\int\sin^{-1}(x)dx=x\sin^{-1}(x)-\int\frac{x}{\sqrt{1-x^2}}dx$ $=x\sin^{-1}(x)-\left(-\frac{1}{2}\right)\int\frac{-2x}{\sqrt{1-x^2}}dx$ $=x\sin^{-1}(x)+\frac{\frac{1}{2}}{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}+1}+c$ $=x\sin^{-1}(x)+\sqrt{1-x^2}+c$ Then $\int\sin^{-1}(f)\cdot f'dx=f\sin^{-1}(f)+\sqrt{1-f^2}+c$ where f is a function in x. $\int\frac{x}{2}\sin^{-1}\left(\frac{x^2}{4}\right)dx=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\sqrt{1-\left(\frac{x^2}{4}\right)^2}+c$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\frac{1}{4}\sqrt{16-x^4}+c$	

6	Solution [6] Abstract Vectors	
	$\mathbf{m}\times\mathbf{n}$ is a vector perpendicular to both $\mathbf{m}$ and $\mathbf{n}$. Thus, $\mathbf{m}\cdot(\mathbf{m}\times\mathbf{n})=0$.	
(a)	$\overline{OM}=\frac{1}{2}(\mathbf{a}+\mathbf{c})$ MR is perpendicular to p, so MR is parallel to $\mathbf{a}\times\mathbf{b}$. $\overline{MR}=\lambda(\mathbf{a}\times\mathbf{b})$, for $\lambda\in\mathbb{R}$ $\overline{OR}-\overline{OM}=\lambda(\mathbf{a}\times\mathbf{b})$ $\overline{OR}=\frac{1}{2}(\mathbf{a}+\mathbf{c})+\lambda(\mathbf{a}\times\mathbf{b})$ Thus, R is a set of points on the line ℓ given by the equation $\mathbf{r}=\frac{1}{2}(\mathbf{a}+\mathbf{c})+\lambda(\mathbf{a}\times\mathbf{b}),\ \lambda\in\mathbb{R}$.	
(b)	$p:\mathbf{r}=\alpha\mathbf{a}+\beta\mathbf{b},\ \alpha,\beta\in\mathbb{R}$. At point of intersection,	

	$\frac{1}{2}(\mathbf{a}+\mathbf{c})+\lambda(\mathbf{a}\times\mathbf{b})=\alpha\mathbf{a}+\beta\mathbf{b}$ (*) Scalar product (*) with **a**, $\frac{1}{2}\lvert\mathbf{a}\rvert^2+\frac{1}{2}\mathbf{a}\cdot\mathbf{c}+\lambda\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\alpha\lvert\mathbf{a}\rvert^2+\beta\mathbf{a}\cdot\mathbf{b}$ $\frac{1}{2}+\frac{1}{2}(-2)=\alpha$ $\alpha=-\frac{1}{2}$ Scalar product (*) with **b**, $\frac{1}{2}\mathbf{b}\cdot\mathbf{a}+\frac{1}{2}\mathbf{b}\cdot\mathbf{c}+\lambda\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=\alpha\mathbf{b}\cdot\mathbf{a}+\beta\lvert\mathbf{b}\rvert^2$ $\frac{1}{2}(4)=\beta$ $\beta=2$ Position vector of the point of intersection is $-\frac{1}{2}\mathbf{a}+2\mathbf{b}$.	
	Alternative Since **a**, **b** are perpendicular unit vectors parallel to plane p, $\mathbf{a}\times\mathbf{b}$ would be perpendicular to **a** and **b** (and thus to p). Thus, $\overrightarrow{OM}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{a}\times\mathbf{b}$ for some $\alpha,\beta,\gamma\in\mathbb{R}$. Since X is the foot of perpendicular from M to p, $\overrightarrow{OX}=\alpha\mathbf{a}+\beta\mathbf{b}$ where $\alpha=\overrightarrow{OM}\cdot\mathbf{a}$ $=\frac{1}{2}(\mathbf{a}+\mathbf{c})\cdot\mathbf{a}$ $=\frac{1}{2}\mathbf{a}\cdot\mathbf{a}+\frac{1}{2}\mathbf{c}\cdot\mathbf{a}$ $=\frac{1}{2}(1)+\frac{1}{2}(-2)$ $=-\frac{1}{2}$	

	$\beta=\overrightarrow{OM}\cdot\mathbf{b}$ $=\frac{1}{2}(\mathbf{a}+\mathbf{c})\cdot\mathbf{b}$ $=\frac{1}{2}\mathbf{a}\cdot\mathbf{b}+\frac{1}{2}\mathbf{c}\cdot\mathbf{b}$ $=\frac{1}{2}(0)+\frac{1}{2}(4)$ $=2$ Position vector of the point of intersection is $-\frac{1}{2}\mathbf{a}+2\mathbf{b}$.	

7	Solution [7] Differentiation and Applications	
(a)	$x^2+y^2-4y=xy$ $2x+2y\frac{dy}{dx}-4\frac{dy}{dx}=y+x\frac{dy}{dx}$ $(2y-x-4)\frac{dy}{dx}+2x-y=0$ $(2y-x-4)\frac{dy}{dx}=y-2x$	
(b)	Area of triangle *OMN*, *A* $=\frac{1}{2}(3)(y)=1.5y$	
(c)	$A=1.5y$ $\frac{dA}{dx}=\frac{dA}{dy}\frac{dy}{dx}$ $=1.5\frac{dy}{dx}$ $=1.5\left(\frac{y-2x}{2y-x-4}\right)$	
(d)	At stationary point, $\frac{dA}{dx}=0$ $1.5\left(\frac{y-2x}{2y-x-4}\right)=0$ $y=2x$ ---(*) $x^2+(2x)^2-4(2x)=x(2x)$ $3x^2-8x=0$	

	Solving, $x=\frac{8}{3}$ or $x=0$ (Rejected since $x=0 \Rightarrow y=0$, using (*)) Note: GC skills to check your answer. $x^2+y^2-4y=xy$ $y^2-(x+4)y+x^2=0$ $y=\frac{(x+4)\pm\sqrt{(x+4)^2-4x^2}}{2}$ $y=\frac{(x+4)\pm\sqrt{(-x+4)(3x+4)}}{2}$ This implies that the maximal set of values of x for $y=\frac{(x+4)\pm\sqrt{(-x+4)(3x+4)}}{2}$ to be valid is $-\frac{3}{4}\le x\le 4$. To find the max value of $A=1.5y$ is equivalent to finding the maximum value of y. Use GC to graph $y=\frac{(x+4)+\sqrt{(-x+4)(3x+4)}}{2}$, to find the coordinates of the maximum point.	

8	Solution [8] Maclaurin's Series	
(a)	$(1+9x^2)\frac{dy}{dx}=3y$ $(1+9x^2)\frac{d^2y}{dx^2}+18x\frac{dy}{dx}=3\frac{dy}{dx}$ $(1+9x^2)\frac{d^2y}{dx^2}+(18x-3)\frac{dy}{dx}=0$	
(b)	$(1+9x^2)\frac{d^2y}{dx^2}+(18x-3)\frac{dy}{dx}=0$ $(1+9x^2)\frac{d^3y}{dx^3}+18x\frac{d^2y}{dx^2}+(18x-3)\frac{d^2y}{dx^2}+18\frac{dy}{dx}=0$ When $x=0$, $y=e^2$, $\frac{dy}{dx}=3e^2$, $\frac{d^2y}{dx^2}=9e^2$, $\frac{d^3y}{dx^3}=-27e^2$ $y=e^2+3e^2x+\frac{9e^2}{2}x^2-\frac{27e^2}{6}x^3+\cdots$ $=e^2+3e^2x+\frac{9e^2}{2}x^2-\frac{9e^2}{2}x^3+\cdots$	
(c)	$\ln y=2+\tan^{-1}3x$ $y=e^{2+\tan^{-1}3x}=e^2e^{\tan^{-1}3x}$ $e^{\tan^{-1}3x}=\frac{y}{e^2}$ $=1+3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\cdots$	

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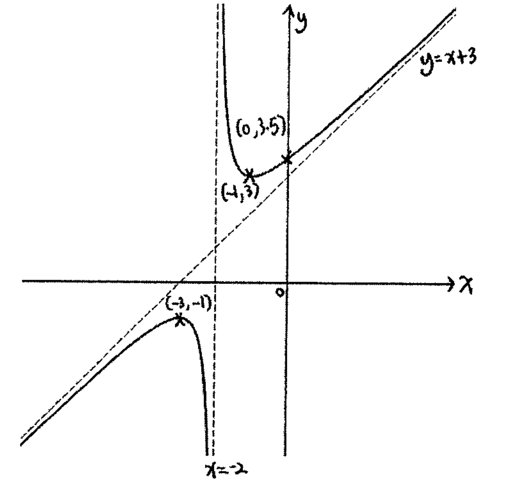
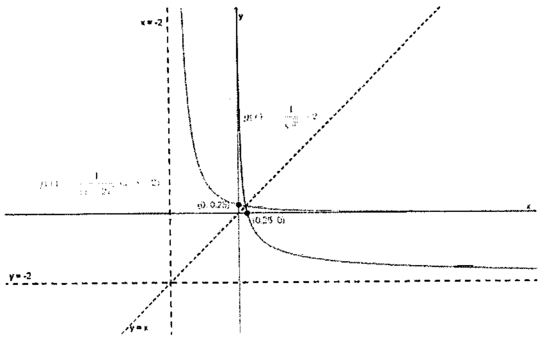
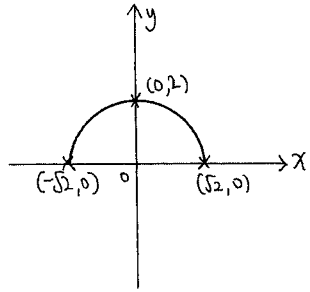
9	Solution [8] Graphing techniques	
(a)	$\frac{dy}{dx} = a - \frac{b-2a}{(x+2)^2}$ At stationary point, $\frac{dy}{dx} = 0$ $a - \frac{b-2a}{(x+2)^2} = 0$ $a = \frac{b-2a}{(x+2)^2}$ $(x+2)^2 = \frac{b-2a}{a}$ Note that $(x+2)^2 \geq 0$ for all x. Hence, $\frac{b-2a}{a} \geq 0$ Since $a > 0$ (given in question), $b-2a \geq 0$ Since $a \neq \frac{1}{2}b$ (given in question), $b-2a > 0$ (or $b > 2a$)	
(b)	$y = x+3+\frac{1}{x+2}$ $y(x+2) = (x+3)(x+2)+1$ $xy+2y = x^2+5x+7$ $x^2+5x-xy+7-2y=0$ $x^2+x(5-y)+7-2y=0$ Find values of y where $b^2-4ac<0$: $b^2-4ac = (5-y)^2-4(1)(7-2y)$ $= 25-10y+y^2-28+8y$ $= y^2-2y-3$ $= (y+1)(y-3)$ $(y+1)(y-3)<0$ $-1<y<3$ Hence y cannot lie between -1 and 3 (shown).	

(c)



10	Solution [9] [Functions[	
(a)	$a=-2$	
(b)	$y=\dfrac{1}{(x+2)^2}$ $\pm\dfrac{1}{\sqrt{y}}=x+2$ $\dfrac{1}{\sqrt{y}}=x+2$ since $x>-2$ $x=\dfrac{1}{\sqrt{y}}-2$ $f^{-1}(x)=\dfrac{1}{\sqrt{x}}-2$ $D_{f^{-1}}=R_f=(0,\infty)$	
(c)		
(d)	$R_f=(0,\infty),\ D_g=[-1,\infty)$ Since $R_f\subseteq D_g$, gf exists. $D_{gf}=(-2,\infty)\xrightarrow{f}(0,\infty)\xrightarrow{g}[a,b)=R_{gf}$ Therefore $R_{gf}=[a,b)$	

11	Solution [10] Integration (Area)		
(a)			
(b)	$y=\sqrt{4-2x^2}$ $\dfrac{dy}{dx}=\dfrac{1}{2}(4-2x^2)^{-\frac{1}{2}}(-4x)$ $=\dfrac{-2x}{\sqrt{4-2x^2}}$ When $x=1,\ \dfrac{dy}{dx}=-\sqrt{2}$ $y-\sqrt{2}=-\sqrt{2}(x-1)$ $y=-\sqrt{2}x+2\sqrt{2}$		
(c)	$\int_1^{\sqrt{2}}\sqrt{4-2x^2}\ dx$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{4-2\left(\sqrt{2}\sin\theta\right)^2}\ \sqrt{2}\cos\theta\ d\theta$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{4\cos^2\theta}\sqrt{2}\cos\theta\ d\theta$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}2\cos\theta\sqrt{2}\cos\theta\ d\theta$ $=2\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\ d\theta$ $=2\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\dfrac{1+\cos 2\theta}{2}\ d\theta$	$x=\sqrt{2}\sin\theta$ $\dfrac{dx}{d\theta}=\sqrt{2}\cos\theta$ When $x=1,\ \theta=\dfrac{\pi}{4}$ When $x=\sqrt{2},\ \theta=\dfrac{\pi}{2}$	

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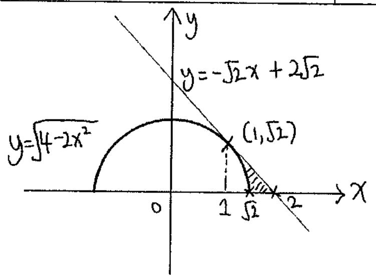
$$= \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1+\cos 2\theta \, d\theta$$

$$= \sqrt{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \sqrt{2}\left[\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \sqrt{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

(d)



Area of region

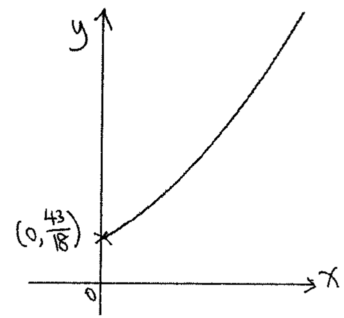
$$= \frac{1}{2}(\sqrt{2})(1) - \sqrt{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\pi + \frac{\sqrt{2}}{2}$$

$$= \sqrt{2} - \frac{\sqrt{2}}{4}\pi \text{ units}^2$$

12 Solution [11] Normal to parametric curve

(a)



(b)

$x = 6t - 5 \qquad y = 2t^2 + 1 \quad \text{where } t \ge \frac{5}{6}$

$\frac{dx}{dt} = 6 \qquad \frac{dy}{dt} = 4t$

$\frac{dy}{dx} = \frac{4t}{6} = \frac{2}{3}t$

When $t = 1$, $\frac{dy}{dx} = \frac{2}{3} \Rightarrow$ gradient of normal at $t = 1$ is $-\frac{3}{2}$

When $t = 1$, $x = 1$ and $y = 3$

$y - 3 = -\frac{3}{2}(x - 1)$

$y - 3 = -\frac{3}{2}x + \frac{3}{2}$

$y = -\frac{3}{2}x + \frac{9}{2}$

Alternatively

$x = 6t - 5 \qquad y = 2t^2 + 1 \quad \text{where } t \ge \frac{5}{6}$

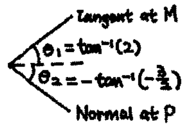
Then $y = \frac{1}{18}(x+5)^2 + 1$, $\frac{dy}{dx} = \frac{1}{9}(x+5)$ where $x \ge 0$

When $t = 1$, then $x = 1$, $y = 3$, $\frac{dy}{dx} = \frac{1}{9}(1+5) = \frac{2}{3}$

Eqn of Normal: $y - 3 = -\frac{3}{2}(x-1) \Rightarrow y = -\frac{3}{2}x + \frac{9}{2}$

(c) Tangent to C at point M, gradient is

$\frac{2}{3}t = \frac{2}{3}(3) = 2$



Note:
- $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$
- $\tan^{-1}(-\frac{3}{2})$ gives a negative value.

$\theta_1 + \theta_2$

$= \tan^{-1}(2) - \tan^{-1}\left(-\frac{3}{2}\right)$

$= 2.089942$ or 119.7°

Required acute angle

$= \pi - 2.089942$ or $180^\circ - 119.7^\circ$

$= 1.05$ or 60.3°

(d) D has parametric form $x - 5 = 6t - 5$ and $y + 3 = 2t^2 + 1$.

Consider

$x - 5 = 6t - 5$

$x = 6t$

Consider

$y + 3 = 2t^2 + 1$

$y = 2t^2 - 2$

D has parametric form: $x = 6t$, $y = 2t^2 - 2$.

(e) Curve C: $x = 6t - 5$ and $y = 2t^2 + 1$, $t \geq \frac{5}{6}$

Curve E: $x = 7u$ and $y = \frac{9}{u}$, $u \neq 0$

Since curves C and E intersect, we equate the x- and y-coordinates respectively.

Equating $x = 6t - 5$ and $x = 7u$:

$6t - 5 = 7u$

$u = \frac{6t-5}{7}$ ---(1)

Equating $y = 2t^2 + 1$ and $y = \frac{9}{u}$:

$2t^2 + 1 = \frac{9}{u}$ ---(2)

Substitute (1) into (2):

$2t^2 + 1 = \frac{9}{\frac{6t-5}{7}}$

$2t^2 + 1 = \frac{63}{6t-5}$

$(2t^2+1)(6t-5) = 63$

$12t^3 - 10t^2 + 6t - 5 = 63$

$12t^3 - 10t^2 + 6t - 68 = 0$

$6t^3 - 5t^2 + 3t - 34 = 0$ (Shown)

$(t-2)(6t^2 + bt + 17) = 0$

Compare t term:

$17 - 2b = 3$

$b = 7$

$(t-2)(6t^2 + 7t + 17) = 0$

$t - 2 = 0$ or $6t^2 + 7t + 17 = 0$

$t = 2$ **(Not Applicable., justification below)**

<u>EITHER (discriminant)</u>

For $6t^2 + 7t + 17$,

discriminant $= b^2 - 4ac$

$= (7)^2 - 4(6)(17)$

$= -359 < 0$

and coefficient of t^2 is positive

$\therefore t = 2$ is the only real root and this implies there is no other point of intersection.

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OR (completing the square)

$$6t^2+7t+17=6\left(t^2+\frac{7}{6}t+\frac{17}{6}\right)$$
$$=6\left[\left(t+\frac{7}{12}\right)^2-\left(\frac{7}{12}\right)^2+\frac{17}{6}\right]$$
$$=6\left(t+\frac{7}{12}\right)^2+\frac{359}{24}>0$$

When $t=2$,
$x=6(2)-5$ and $y=2(2)^2+1$
$x=7$ and $y=9$

The point of intersection is $A(7,9)$.

13 Solution [12] Differential equations

(a) Let $\frac{dA}{dt}$ and $\frac{dB}{dt}$ be the rate of roasting of coffee beans and rate of coffee beans sold respectively.

$$\frac{dA}{dt}=10,\quad \frac{dB}{dt}\propto x^2 \quad\Rightarrow\quad \frac{dB}{dt}=kx^2$$
$$\frac{dx}{dt}=\frac{dA}{dt}-\frac{dB}{dt}=10-kx^2$$

When $x=5$, $\frac{dx}{dt}=0$:
$$0=10-k(5)^2$$
$$25k=10$$
$$k=\frac{2}{5}$$
$$\therefore \frac{dx}{dt}=10-\frac{2}{5}x^2$$

$$\frac{dx}{dt}=10-\frac{2}{5}x^2$$
$$=-\frac{2}{5}\left(x^2-25\right)$$

$$\int\frac{1}{x^2-25}\,dx=\int-\frac{2}{5}\,dt$$
$$\frac{1}{2(5)}\ln\left|\frac{x-5}{x+5}\right|=-\frac{2}{5}t+c$$

(b) Let $\frac{dC}{dt}$ and $\frac{dB}{dt}$ be the new rate of roasting of coffee beans and rate of coffee beans sold respectively.

$$\frac{dC}{dt}=3x,\quad \frac{dB}{dt}\propto x^2 \quad\Rightarrow\quad \frac{dB}{dt}=kx^2$$
$$\frac{dx}{dt}=\frac{dC}{dt}-\frac{dB}{dt}=3x-kx^2$$

When $x=6$, $\frac{dx}{dt}=6$:

$$6 = 3(6) - k(6)^2$$
$$6 = 18 - 36k$$
$$36k = 12$$
$$k = \frac{1}{3}$$
$$\therefore \frac{dx}{dt} = 3x - \frac{1}{3}x^2$$

(c)

$$\frac{dx}{dt} = 3x - \frac{1}{3}x^2$$
$$= \frac{1}{3}x(9-x)$$
$$\int \frac{1}{x(9-x)}\,dx = \int \frac{1}{3}\,dt$$
$$\int \frac{1}{9}\left(\frac{1}{x} + \frac{1}{9-x}\right)dx = \int \frac{1}{3}\,dt$$
$$\int \frac{1}{x} + \frac{1}{9-x}\,dx = \int 3\,dt$$
$$\ln|x| - \ln|9-x| = 3t + c$$
$$\ln\left|\frac{x}{9-x}\right| = 3t + c$$
$$\left|\frac{x}{9-x}\right| = e^{3t+c}$$
$$\frac{x}{9-x} = Ae^{3t}, \text{ where } A = \pm e^{c}$$
$$x = Ae^{3t}(9-x)$$
$$x = 9Ae^{3t} - xAe^{3t}$$
$$x + xAe^{3t} = 9Ae^{3t}$$
$$x(1 + Ae^{3t}) = 9Ae^{3t}$$
$$x = \frac{9Ae^{3t}}{1 + Ae^{3t}}$$

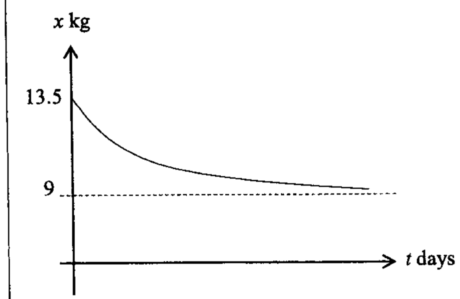
When $t = 0$, $x = 13.5$:

$$x = \frac{9Ae^{3t}}{1 + Ae^{3t}}$$
$$13.5 = \frac{9A}{1+A}$$
$$13.5(1+A) = 9A$$
$$13.5 + 13.5A = 9A$$
$$4.5A = -13.5$$
$$A = -3$$

$$x = \frac{9(-3)e^{3t}}{1 - 3e^{3t}}$$
$$x = \frac{-27e^{3t}}{1 - 3e^{3t}}$$
$$x = \frac{27e^{3t}}{3e^{3t} - 1} \text{ or } x = \frac{27}{3 - e^{-3t}}$$

As $t \to \infty$, $x = \frac{27e^{3t}}{3e^{3t} - 1} = \frac{27}{3 - e^{-3t}} \to 9$

In the long term, the amount of roasted coffee beans remaining in the cafe will stabilise at 9 kg/approaches 9 kg.



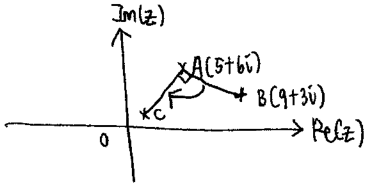
(d) There is always roasted coffee beans left in the inventory for selling.
OR
The amount of roasted coffee beans stabilizing at a particular value allows the owner to plan for storage of adequate capacity.

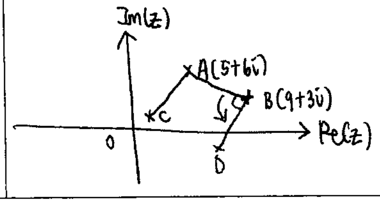
BP~554

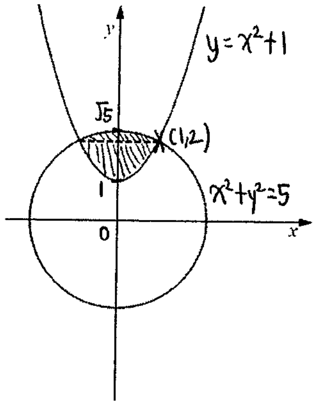
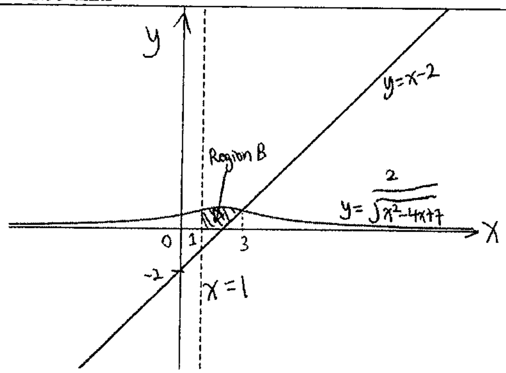
2025 H2MA Prelim Paper 2

1	Solution [5] Sequences	
(a) (i)	For $N=4$, the sequence takes on a **constant value** with $u_n = 4$ for all $n \geq 1$.	
(a) (ii)	For $N=3$, $u_1 = 3$ $u_2 = 3(3) - 8 = 1$ $u_3 = 3(-5) - 8 = -23$ ... The sequence **decreases** and **approaches negative infinity**.	
(b)	**Method 1** $u_5 = 409$ Using $u_{n+1} = 3u_n - 8$, $3u_4 - 8 = 409 \Rightarrow u_4 = 139$ $3u_3 - 8 = 139 \Rightarrow u_3 = 49$ $3u_2 - 8 = 49 \Rightarrow u_2 = 19$ $3u_1 - 8 = 19 \Rightarrow u_1 = 9 = N$ **Method 2** $u_1 = N$ $u_2 = 3N - 8$ $u_3 = 9N - 32$ $u_4 = 27N - 104$ $u_5 = 81N - 320$ $81N - 320 = 409$ $N = 9$	
(c)	From part (a), when $u_1 = 4$, $u_n = 4$ for $n \geq 1$ $v_n = (-1)^n (u_n + k)$ $v_1 = -(4+k)$ $v_2 = (4+k)$ $v_3 = -(4+k)$ $\vdots$	

	For $v_n = (-1)^n (u_n + k)$ to be convergent, then $k = -4$.	

2	Solution [5] Complex Numbers	
(a)	The line segment AC can be obtained by rotating the line segment AB, 90° in a clockwise direction about A. $z_C - z_A = -\mathrm{i}(z_B - z_A)$ $z_C = z_A - \mathrm{i}(z_B - z_A)$ $z_C = (5+6\mathrm{i}) - \mathrm{i}[(9+3\mathrm{i})-(5+6\mathrm{i})]$ $= (5+6\mathrm{i}) - \mathrm{i}[4-3\mathrm{i}]$ $= 2+2\mathrm{i}$ 	
(b)	**Method 1** $ABDC$ is a parallelogram Midpoint of AD = Midpoint of BC $\frac{z_A+z_D}{2} = \frac{z_B+z_C}{2}$ $(5+6\mathrm{i})+z_D = (9+3\mathrm{i})+(2+2\mathrm{i})$ $z_D = 11+5\mathrm{i}-(5+6\mathrm{i})$ $z_D = 6-\mathrm{i}$	
	Method 2 $\overrightarrow{BD} = \overrightarrow{AC}$ $z_D - z_B = z_C - z_A$ $z_D = z_B + z_C - z_A$ $z_D = 6-\mathrm{i}$	
	Method 3 BD can be obtained by rotating the line segment BA, 90° in an anti-clockwise direction about B.	

	$\mathrm{i}z_{BA} = z_{BD}$ $\mathrm{i}(z_B - z_A) = z_D - z_B$ $\mathrm{i}(5+6\mathrm{i}-9-3\mathrm{i}) = z_D - 9 - 3\mathrm{i}$ $\mathrm{i}(3\mathrm{i}-4) = z_D - 9 - 3\mathrm{i}$ $z_D = 9+3\mathrm{i}+\mathrm{i}(3\mathrm{i}-4)$ $z_D = 6-\mathrm{i}$ 	

3	Solution [8] Volume of revolution	
(a)	 Volume required $=\pi\int_1^2 y-1\,dy+\pi\int_2^{\sqrt{5}} 5-y^2\,dy$ $=1.95$ units^3	
(b)	 x-coordinate of point of intersection between $y=\frac{2}{\sqrt{x^2-4x+7}}$ and $y=x-2$ is $x=3$ (from GC)	

Volume required

$$=\pi\int_1^3\left(\frac{2}{\sqrt{x^2-4x+7}}\right)^2 dx-\pi\int_2^3(x-2)^2\,dx$$

OR $\pi\int_1^3\left(\frac{2}{\sqrt{x^2-4x+7}}\right)^2 dx-\frac{1}{3}\pi(1)^2(1)$

$$=\pi\int_1^3\frac{4}{x^2-4x+7}dx-\pi\int_2^3 x^2-4x+4\,dx$$

$$=4\pi\int_1^3\frac{1}{(x-2)^2-4+7}dx-\pi\left[\frac{x^3}{3}-2x^2+4x\right]_2^3$$

$$=4\pi\int_1^3\frac{1}{(x-2)^2+\left(\sqrt{3}\right)^2}dx-\pi\left[\frac{x^3}{3}-2x^2+4x\right]_2^3$$

$$=4\pi\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)\right]_1^3-\frac{1}{3}\pi$$

$$=\frac{4}{\sqrt{3}}\pi\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]-\frac{1}{3}\pi$$

$$=\frac{4}{\sqrt{3}}\pi\left[\frac{\pi}{6}+\frac{\pi}{6}\right]-\frac{1}{3}\pi$$

$$=\frac{4}{\sqrt{3}}\pi\left(\frac{\pi}{3}\right)-\frac{1}{3}\pi$$

$$=\frac{4}{9}\sqrt{3}\pi^2-\frac{1}{3}\pi \text{ units}^3$$

$a=\frac{4}{9}$, $b=-\frac{1}{3}$

4 Solution [10] AP GP

(a) Let u_n be the dose on the n^{th} day.

$$u_n = 0.75 + 0.5(n-1)$$

$$S_n = \frac{n}{2}\left(2(0.75)+(n-1)(0.5)\right)$$

$$10 \le \frac{n}{2}\left(1+\frac{n}{2}\right) \qquad \text{----(*)}$$

$$\frac{1}{4}n^2 + \frac{1}{2}n \ge 10$$

$$n^2 + 2n \ge 40$$

$$(n+1)^2 \ge 41$$

$$n \ge -1+\sqrt{41} \text{ or } n \le -1-\sqrt{41}$$

(rej as no. of days cannot be -ve)

$$n \ge 5.40312$$

Therefore, it takes 6 days.

Alternatively,

When $n=5$, $\frac{1}{4}n^2+\frac{1}{2}n = 8.75 < 10$

When $n=6$, $\frac{1}{4}n^2+\frac{1}{2}n = 12 \ge 10$

Therefore, it takes 6 days.

$u_6 = 0.75+0.5(5) = 3.25$

The dose on the 6th day is 3.25 units.

(b)(i) Increase on n^{th} day

$$= S_n - S_{n-1}$$

$$= \frac{3}{20}\left(\frac{5^n-4^n}{5^{n-1}}\right) - \frac{3}{20}\left(\frac{5^{n-1}-4^{n-1}}{5^{n-2}}\right)$$

$$= \frac{3}{20}\left(\frac{5^n-4^n}{5^{n-1}} - \frac{5^{n-1}-4^{n-1}}{5^{n-2}}\right)$$

$$= \frac{3}{20}\left(\frac{1}{5^{n-2}}\right)\left(5^{n-1} - \frac{4}{5}(4^{n-1}) - 5^{n-1} + 4^{n-1}\right)$$

$$= \frac{3}{20}\left(\frac{1}{5^{n-2}}\right)\left(\frac{1}{5}\right)\left(4^{n-1}\right)$$

$$= \frac{3}{20}\left(\frac{4}{5}\right)^{n-1}$$

which is of the form ar^{n-1}, where $a = \frac{3}{20}$ and $r = \frac{4}{5}$ which is a GP.

Alternative:

Consider $\frac{u_{n+1}}{u_n} = \dfrac{\frac{3}{20}\left(\frac{4}{5}\right)^{n+1-1}}{\frac{3}{20}\left(\frac{4}{5}\right)^{n-1}} = \frac{4}{5}$ which is a constant

Since consecutive terms have a common ratio $r = \frac{4}{5}$, the increase in dosage follows a GP.

(b)(ii) **Method 1**

Long term dosage

$$= \lim_{n\to\infty} \frac{3}{20}\left(\frac{5^n-4^n}{5^{n-1}}\right)$$

$$= \frac{3/20}{1-4/5}$$

$$= \frac{3}{4}$$

In the long term, the total dosage approaches 0.75 units.

Note:

After n days, the total dosage of the medicine is given by

	$\frac{3}{20}\left(\frac{5^n-4^n}{5^{n-1}}\right)$. Let S_n denote the total dosage of the new medicine after n days. Then $S_n=\frac{3}{20}\left(\frac{5^n-4^n}{5^{n-1}}\right)$. Let v_n the change in dosage on nth day found in part (b)(i) where $v_n=S_n-S_{n-1}$ Since v_n is a geometric progression, we can use $S_\infty=\frac{a}{1-r}$, the sum to infinity formula to find the long term total dosage.	
	<u>Method 2</u> $\frac{3}{20}\left(\frac{5^n-4^n}{5^{n-1}}\right)$ $=\frac{3}{20}(5)\left(\frac{5^n-4^n}{5^n}\right)$ $=\frac{3}{20}(5)\left(1-\left(\frac{4}{5}\right)^n\right)$ As $n\to\infty$, $\left(\frac{4}{5}\right)^n\to 0$ Total dosage in the long term $=\frac{3}{20}(5)(1-0)$ $=\frac{3}{4}$	
(c)	If the dosage increases as an AP, then it will continue to **<u>increase to</u>** (positive) **<u>infinity</u>**.	

5	Solution [12] 3D Vectors	
(a)	Since $\begin{pmatrix}1\\-1\\5\end{pmatrix}\bullet\begin{pmatrix}-2\\3\\1\end{pmatrix}=-2-3+5=0$, the satellite's path is parallel to Earth's Orbital plane. Since $\begin{pmatrix}3\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\5\end{pmatrix}=3-2+15=16\neq 0$, the satellite does not sit on Earth's Orbital plane. Therefore, the satellite does not cross Earth's Orbital Plane. **<u>Alternatively</u>** Earth: $\mathbf{r}\bullet\begin{pmatrix}1\\-1\\5\end{pmatrix}=0$ ---(1) Satellite: $\mathbf{r}=\begin{pmatrix}3\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\3\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$ ---(2) Sub (2) into (1): $\left[\begin{pmatrix}3\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\3\\1\end{pmatrix}\right]\bullet\begin{pmatrix}1\\-1\\5\end{pmatrix}=0$ ---(3) $16+0\lambda=0$ There is no consistent value of λ for eqn (3) Therefore, the satellite does not cross Earth's Orbital Plane.	

(b)	$l_E : \mathbf{r} = \begin{pmatrix}15\\20\\-1\end{pmatrix} + \mu\begin{pmatrix}4\\-1\\-1\end{pmatrix}, \mu \in \mathbb{R}$ $l_A : \dfrac{x-10}{2} = y+2 = \dfrac{0.25-z}{k}$ $\mathbf{r} = \begin{pmatrix}10\\-2\\0.25\end{pmatrix} + t\begin{pmatrix}2\\1\\-k\end{pmatrix}$, where $t \in \mathbb{R}$ Consider $\begin{pmatrix}10+2t\\-2+t\\0.25-tk\end{pmatrix} = \begin{pmatrix}15+4\mu\\20-\mu\\-1-\mu\end{pmatrix}$ $2t-4\mu=5$ $t+\mu=22$ $\mu-tk=-1.25$ From GC, $t=15.5$, $\mu=6.5$, $tk=7.75$ $\Rightarrow k=\dfrac{7.75}{15.5}=0.5$ Therefore, the paths intersect when $k=0.5$. Therefore, for the paths to not intersect, $k \neq 0.5$, $k \in \mathbb{R}^+$.	
(c)	Let θ be the acute angle between the paths of the Earth and the asteroid. $\cos\theta = \left\lvert \begin{pmatrix}2\\1\\-3\end{pmatrix} \bullet \begin{pmatrix}4\\-1\\-1\end{pmatrix} \right\rvert$ $= \dfrac{\lvert 8-1+3 \rvert}{\sqrt{2^2+1^2+(-3)^2}\sqrt{4^2+(-1)^2+(-1)^2}}$ $= \dfrac{10}{\sqrt{14}\sqrt{18}}$ $\theta = \cos^{-1}\left(\dfrac{10}{\sqrt{14}\sqrt{18}}\right) = 51.0°$ (1 d.p.)	

(d) (i)	Distance $= \left\lvert \begin{pmatrix}12\\18\\-3\end{pmatrix} - \begin{pmatrix}12\\-1\\-2.75\end{pmatrix} \right\rvert$ $= \left\lvert \begin{pmatrix}0\\19\\-0.25\end{pmatrix} \right\rvert$ $= \sqrt{19^2+(-0.25)^2}$ $= 19.0$ million km (3 s.f.)	
(d) (ii)	The shortest distance between the paths will be when the vector connecting the two points on the paths is perpendicular to each of the two paths. $\begin{pmatrix}0\\19\\-0.25\end{pmatrix} \bullet \begin{pmatrix}2\\1\\-3\end{pmatrix} = 19.75 \neq 0$ ---(1) OR $\begin{pmatrix}0\\19\\-0.25\end{pmatrix} \bullet \begin{pmatrix}4\\-1\\-1\end{pmatrix} = -18.75 \neq 0$---(2) Since the vector is not perpendicular to one of the paths, the distance found is not the shortest distance between the paths. To show that $\left\lvert \begin{pmatrix}0\\19\\-0.25\end{pmatrix} \right\rvert$ is not the shortest distance, it suffices to show either (1) or (2).	

BP~560

6 Solution [4] Permutations and Combinations

(a) Number of ways

$(5-1)!=24$

(b) <u>Case 1:</u> One person gets 3 more slices

$\binom{3}{1}\times\left(\frac{5!}{3!}\right)=60$

<u>Case 2:</u> Two people get 2 more slices

$\binom{3}{2}\times\left(\frac{5!}{2!2!}\right)=90$

Total Number of ways = 150

Alternatively

<u>Case 1:</u> One person gets 3 more slices

$\binom{3}{1}\binom{5}{3}\binom{2}{1}=60$

<u>Case 2:</u> Two people get 2 more slices

$\binom{3}{2}\binom{5}{2}\binom{3}{2}=90$

Total Number of ways = 150

7 Solution [5] DRV

(a) Let X be the r.v. denoting the score.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$

$P(X=1)=\left(\frac{1}{6}+\frac{1}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)=\frac{1}{18}$

$P(X=2)=\left(\frac{1}{6}+\frac{1}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)=\frac{1}{18}$

$P(X=3)=\frac{1}{6}+\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)=\frac{4}{18}$ or $\frac{2}{9}$

$P(X=4)=\frac{1}{6}+\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)=\frac{4}{18}$ or $\frac{2}{9}$

$P(X=5)=\frac{1}{6}+\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)=\frac{4}{18}$ or $\frac{2}{9}$

$P(X=6)=\frac{1}{6}+\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)=\frac{4}{18}$ or $\frac{2}{9}$

(b)

$$E(X)=(1)\left(\frac{1}{18}\right)+(2)\left(\frac{1}{18}\right)+(3)\left(\frac{4}{18}\right)+4\left(\frac{4}{18}\right)+(5)\left(\frac{4}{18}\right)+(6)\left(\frac{4}{18}\right)$$

$$=\frac{1+2+12+16+20+24}{18}$$

$$=\frac{75}{18}=\frac{25}{6}$$

$\frac{25}{6}$ is the **average** score when the game is played **infinitely many (or a large number of) times.**

(c)	$E(X^2)=\frac{349}{18}$ $E(X)=\frac{25}{6}$ $\text{Var}(X)=\frac{349}{18}-\left(\frac{25}{6}\right)^2=\frac{73}{36}$	

8	Solution [7] Probability	
(a)	$P(W,G,G)=\frac{3!}{2!}\left(\frac{1}{10}\right)\left(\frac{3}{10}\right)^2=\frac{27}{1000}$	
(b)	P(White ball drawn | Red ball on 2nd attempt) $=\frac{\text{P(White ball drawn} \cap \text{Red ball on 2nd attempt)}}{\text{P(Red ball on 2nd attempt)}}$ ---(*) $=\frac{P(WRW')+P(RRW)+P(BRW)+P(GRW)}{P(R'R)+P(RR)}$ $=\frac{\left[\left(\frac{1}{10}\right)\left(\frac{4}{10}\right)\left(\frac{8}{9}\right)+\left(\frac{4}{10}\right)\left(\frac{3}{9}\right)\left(\frac{1}{8}\right)+\left(\frac{2}{10}\right)\left(\frac{4}{10}\right)\left(\frac{1}{9}\right)+\left(\frac{3}{10}\right)\left(\frac{4}{10}\right)\left(\frac{1}{9}\right)\right]}{\left(\frac{6}{10}\right)\left(\frac{4}{10}\right)+\left(\frac{4}{10}\right)\left(\frac{3}{9}\right)}$ $=\frac{67/900}{28/75}$ $=\frac{67}{336}$	
(c)	$P(A)=\frac{1}{10}$ $P(B)=P(RW)+P(R'W)=\left(\frac{4}{10}\right)\left(\frac{1}{9}\right)+\left(\frac{6}{10}\right)\left(\frac{1}{10}\right)=\frac{47}{450}$ $P(A\cap B)=P(WW)=\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)=\frac{1}{100}$ $P(A\cap B)\neq P(A)P(B)$ Therefore, A and B are **not** independent. **<u>Alternative</u>**: $P(A)=\frac{1}{10}$ $P(B)=P(RW)+P(R'W)=\left(\frac{4}{10}\right)\left(\frac{1}{9}\right)+\left(\frac{6}{10}\right)\left(\frac{1}{10}\right)=\frac{47}{450}$ $P(B\mid A)=\left(\frac{1}{10}\times\frac{1}{10}\right)\div\left(\frac{1}{10}\right)\neq\frac{47}{450}$ $P(B\mid A)\neq P(B)$ Therefore, A and B are **not** independent.	

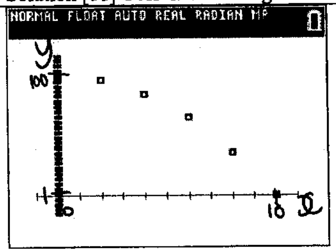
BP~562

9	Solution [8] Binomial Distribution	
(a)	Assume that the probability in which a lens is defective is constant throughout the entire population of lenses produced. This may not be valid, since machine parts degrade gradually which affects the manufactering process and thus probability of a lens being defective changes over time.	
OR	Assume that the event in which a lens is defective is independent to all other lenses. This may not be appropriate, since it is possible that a defective lens could cause other lens to be defective downstream (e.g. a defective lens occurring during the process of production causes misalignment in subsequent lenses and thus causes more defective lenses).	
(b)	Let X denote the number of defective lenses, out of 100. $X \sim \mathrm{B}\left(100, \frac{1}{20}\right)$ $\mathrm{P}(X=3)=0.13958$ Let Y be the random variable for the number of samples that has exactly 3 defective lenses, out of 30. $Y \sim \mathrm{B}(30, 0.13958)$ $\mathrm{P}(Y>10)$ $=1-\mathrm{P}(Y \le 10)$ $=0.00163393$ $=0.00163$	

(c)	$\mathrm{P}(X_1+X_2=10 \mid X_1=3 \text{ or } X_2=3 \text{ but not both})$ $=\dfrac{\mathrm{P}(X_1+X_2=10 \cap (X_1=3 \text{ or } X_2=3))}{\mathrm{P}(X_1=3 \text{ or } X_2=3 \text{ but not both})}$ $=\dfrac{\mathrm{P}(X_1=3 \cap X_2=7)+\mathrm{P}(X_2=3 \cap X_1=7)}{\mathrm{P}(X_1=3)+\mathrm{P}(X_2=3)-2\mathrm{P}(X_1=3)\mathrm{P}(X_2=3)}$ $=\dfrac{(0.13958)(0.10603)\times 2}{0.13958+0.13958-2(0.13958)^2}$ $=0.12323$ $=0.123$ Denominator can be computed via $\mathrm{P}(X_1=3 \text{ or } X_2=3 \text{ but not both})$ $=\mathrm{P}(\text{exactly one of } X_1=3 \text{ or } X_2=3)$ $=P(Y=1)=0.2401883$, where Y is the number of samples, out of 2, to have exactly 3 defective lenses, $Y \sim B(2, 0.13958)$	
(d)	$\dfrac{X_1+X_2}{2} \le 2 \Rightarrow \dfrac{3+X_2}{2} \le 2 \Rightarrow X_2 \le 1$ Required Probability $=\mathrm{P}(X_1 \le 2)+\mathrm{P}(X_1=3)(X_2 \le 1)$ $=0.11826+(0.13958)(0.0370815)$ $=0.123$	

10 Solution [11] Correlation & regression

(a)



From the scatter diagram, **when x increases, y decreases at an increasing rate**, indicating that model B is not suitable. **Model C is also not valid as it is not defined at $x = 10$**.
Thus Model A is the most suitable.

(b)

$r = -0.99995$

$y = -0.99175x^2 + 100.03088$

$y = -0.992x^2 + 100$ (3 s.f)

(c)

When $y = 20$, $x = 8.983 \approx 8.98$ m (to 3 s.f)

The estimate is reliable as $y = 20$ lies within the input data range $1 \le y \le 100$ AND $|r|$ is very close to 1.

(d)(i)

As residuals may be positive or **negative** (or zero), they may **cancel out** if we simply sum the residuals. Squaring the residuals allows us to sum up non-negative numbers for a more accurate assessment of the fit of the model.

(d)(ii)

x	10	20	30	40	50	60
t	29.1	58.3	87.5	116.6	145.8	174.9
$t = 2.916x$	29.16	58.32	87.48	116.64	145.8	174.96

By GC, Sum of squares of residuals
$= (-0.06)^2 + (-0.02)^2 + (0.02)^2 + (-0.04)^2 + 0^2 + (-0.06)^2$
$= 0.0096$

(d)(iii)

The sum of squares of residuals for the line $t = 2.916x + D$ will be less than the sum of squares of residuals for the line $t = 2.916x$ i.e. less than 0.0096.

11 Solution [12] Hypothesis testing

(a)

Each of the 10 000 strings should have an equal probability to be selected as part of the 40 strings to be measured.

The selection of a string is independent of the selection of any other strings.

(b)

Unbiased estimate of population mean,

$\bar{x} = \frac{10.316}{40} + a = a + 0.2579$ cm

$s^2 = \frac{1}{40-1}\left[41.7 - \frac{(10.316)^2}{40}\right]$

$= 1.0010129$ cm^2

(c)

Let X be the random variable denoting the length of the strings,
Let μ be the population mean length of the strings.

To test $H_0 : \mu = 100$
against $H_1 : \mu \ne 100$
at 10% level of significance.

Test Statistics:

Under H_0, $\bar{X} \sim N\left(100, \frac{s^2}{40}\right)$ approximately,

$Z = \frac{\bar{X} - 100}{s/\sqrt{40}} \sim N(0,1)$, where $s = \sqrt{1.0010129}$

Critical region:
Reject H_0 if $z_{cal} < -1.6448536$ or $z_{cal} > 1.6448536$.
$z_{cal} < -1.64$ or $z_{cal} > 1.644$ (3 s.f.)

Since H_0 is not rejected,
$-1.6448536 \le z_{cal} \le 1.6448536$

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$$-1.6448536 \le \frac{\bar{x}-100}{\sqrt{1.0010129}/\sqrt{40}} \le 1.6448536$$

$-0.26020587 \le \bar{x}-100 \le 0.26020587$

$99.73979413 \le \bar{x} \le 100.26020587$

$99.74 \le \bar{x} \le 100.26$

Alternative:
Let X be the random variable denoting the length of the strings,
Let μ be the mean length of the strings.

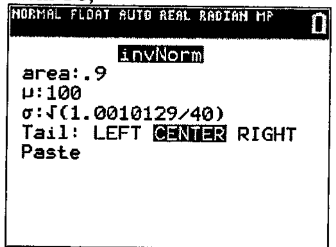
To test $H_0: \mu = 100$
against $H_1: \mu \neq 100$
at 10% level of significance.

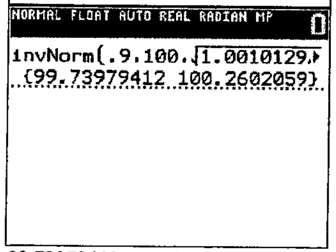
Test Statistics:

Under H_0, $\bar{X} \sim N\left(100, \frac{s^2}{40}\right)$ approximately

$P(a < \bar{X} < b) = 0.9$

From GC,





$99.73979413 < \bar{x} < 100.26020587$

$99.74 \le \bar{x} \le 100.26$

(d) As the sample size is large (n = 40), Central Limit Theorem can be used to approximate the distribution of $\bar{X}$ to a Normal Distribution.
Thus no assumption is required about the distribution of the strings.

(e) For the same set of data,

p-value of 1 tail test $= \frac{1}{2} \times p$-value of 2 tail test

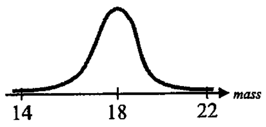
To test $H_0: \mu = 100$
against $H_1: \mu < 100$
at 6% level of significance.

Test Statistics:

Under H_0, p-value $= \frac{1}{2} \times 0.10286 = 0.05143 < 0.06$,

Therefore we reject H_0.
There is sufficient evidence that the strings are shorter than 100cm at 6% level of significance.

BP-566

12	Solution [12] Normal Dist P2 Q12	
(a)		
(b)	Let D be the random variable denoting the mass of a randomly chosen metal disc. $D \sim N(18, 0.6^2)$ Probability required $= \binom{2}{1}[P(D<19)][P(D>19)]$---(*) $= 0.0910128294$ ≈ 0.0910 (3 s.f.)	
(c)	$P(18-k<D<18+k)<0.9$ Since $P(17.01308782<D<18.98691218)=0.9$, then $18+k<18.98691218$ $k<0.987$	
(d)	7.65% of the metal bars have masses more than 21 grams. Let B be the random variable denoting the mass of a randomly chosen metal bar. $B \sim N(20, \sigma^2)$ $P(B>21)=0.0765$ $P\left(Z>\frac{21-20}{\sigma}\right)=0.0765$ $P\left(Z>\frac{1}{\sigma}\right)=0.0765$ $\frac{1}{\sigma}=1.429014729$ $\sigma=0.6997$ $\sigma=0.700$ (3.s.f)	
(e)	Let S be the random variable denoting the mass of a randomly chosen metal strip. $S=0.8B \sim N(0.8\times 20, 0.8^2\times 0.5^2)$	

	$S \sim N(16, 0.4^2)$ $P(15.8<S<17.2)=0.690125006$ Let X be the random variable denoting the number of metal strips whose mass lies between 15.8 grams and 17.2 grams, out of 8 bars. $X \sim B(8, 0.690125006)$ $P(X\geq 5)$ $=1-P(X\leq 4)$ $=0.7875986163$ ≈ 0.788 (3.s.f)	
(f)	Let C be the random variable denoting the mass of a randomly chosen protective case. $C \sim N(0.3\times 20, 0.3^2\times 0.5^2)$ $C \sim N(6, 0.15^2)$ $B \sim N(20, 0.5^2)$ Let $Y=B+C$ $Y \sim N(20+6, 0.5^2+0.15^2)$ $Y \sim N(26, 0.2725)$ Let $T=Y_1+Y_2+Y_3+Y_4$ $\text{Var}(T)=4\times 0.2725$ $=1.09$ (3 s.f)	
	Common Mistake Let C be the random variable denoting the mass of a randomly chosen protective case. $C=0.3B$ $B \sim N(20, 0.5^2)$ Let $Y=B+C$ $Y=B+0.3B=1.3B$ $Y \sim N(1.3\times 20, 1.3^2\times 0.5^2)$ $Y \sim N(26, 0.4225)$ Let $T=Y_1+Y_2+Y_3+Y_4$ $T \sim N(4\times 26, 4\times 0.4225)$ $T \sim N(104, 1.69)$	
