Candidate Name: ______________________ Class: ____________

PIONEER JUNIOR COLLEGE
While I Live, I Learn

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS
Paper 1

9758/01
13 Sept 2017
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 The first three terms of a sequence are given by $u_1 = 70$, $u_2 = 136$, $u_3 = 198$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [4]

2 A sequence $u_0, u_1, u_2, \ldots$ is given by $u_0 = \frac{3}{2}$ and $u_n = u_{n-1} + 2^n - n$ for $n \geq 1$.

(i) Find u_1, u_2 and u_3. [3]

(ii) By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. [5]

3 By sketching the graphs of $y = e^{2x}$ and $y = 2e^{-x} - 1$, solve the inequality

$$e^{2x} \geq 2e^{-x} - 1.$$ [3]

Hence, without using a calculator, find

$$\int_{-1}^{2} \left| e^{2x} - 2e^{-x} + 1 \right| dx,$$

giving your answer in terms of e. [4]

4 The function f is defined by

$$f : x \mapsto \left| \frac{2x+6}{4-x} \right|, \qquad x \in \mathbb{R}, \quad x \neq 4.$$

(i) Sketch the graph of $y = f(x)$, giving the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. Hence state the range of f. [3]

(ii) Determine whether the function f^2 exists, justifying your answer. [1]

(iii) The function f^{-1} exists if the domain of f is further restricted to $x \leq k$. State the greatest value of k. [1]

(iv) Using the domain in (iii), find $y = f^{-1}(x)$ and state the domain of f^{-1}. [4]

5 A curve is given parametrically by the equations

$$x = 2t - 1, \quad y = \frac{1}{2t+1},$$

where $t \in \mathbb{R}$, $t \neq -\frac{1}{2}$.

(i) Sketch the curve, labelling the axial intercepts and asymptotes. [2]

(ii) Find the equation of the tangent to the curve at the point $P(-1, 1)$. [3]

(iii) State the range of values of m for which the line $y = mx$ does not intersect the curve. [1]

(iv) The normal to the curve at P meets the curve again at Q. Find the coordinates of Q. [4]

6 Two expedition teams are to climb a vertical distance of 8500 m from the foot to the peak of a mountain over a period of time.

(i) Team A plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Find the number of days required for Team A to reach the peak. [2]

(ii) Team B plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day. On which day will Team A overtake Team B? [3]

(iii) Explain why Team B will never be able to reach the peak. [2]

(iv) At the end of the 15^{th} day, Team B decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer. [5]

7 The curve C has equation $y = 2 + \dfrac{x-3}{(x-2)(x+1)}$.

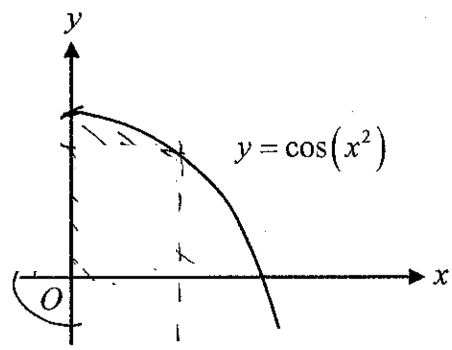
(i) Find algebraically the set of values that y can take. [5]

(ii) Sketch C, giving the coordinates of the axial intercepts, turning points and equations of any asymptotes. [3]

(iii) By adding an appropriate graph to the sketch of C, determine the range of values of k such that the equation $(x-2)^2 + \dfrac{(x-3)^2}{(x-2)^2(x+1)^2} = k^2$ has at least one negative real root. [4]

8 **(a)** Find $\displaystyle\int \sqrt{\frac{1-x}{x}}\,\mathrm{d}x$ by using the substitution $x = \sin^2\theta$, where $0 < \theta < \dfrac{\pi}{2}$. [6]

(b) The diagram below shows a sketch of part of the curve $y = \cos\left(x^2\right)$.



Find the exact volume of the solid generated when the region bounded by the curve $y = \cos\left(x^2\right)$, the axes and the line $x = \dfrac{\sqrt{\pi}}{2}$ is rotated through 2π radians about the y-axis. [7]

9

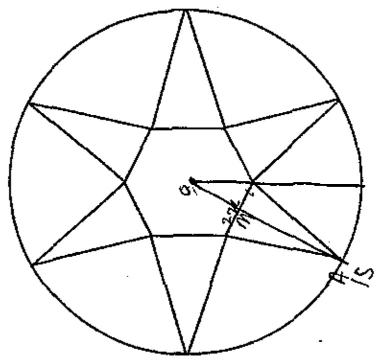


Fig. 1

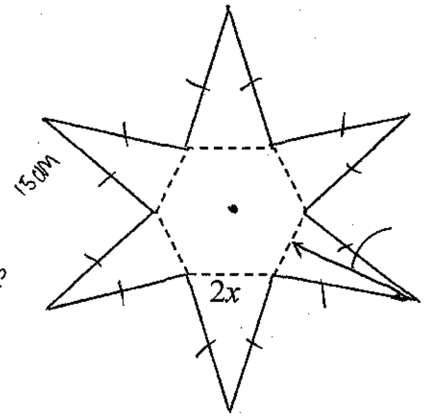


Fig. 2

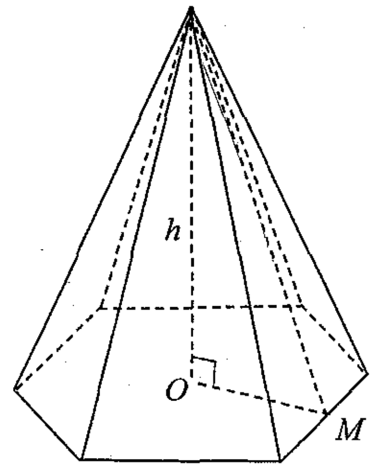


Fig. 3

Fig. 1 shows a piece of circular card of radius 15 cm. A star shape, which consists of a regular hexagon of side $2x$ cm and 6 isosceles triangles, is cut out from the card to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines to form a pyramid of height h cm as shown in Fig. 3.
(The diagrams are not drawn to scale).

(i) By considering triangle AOM as shown in Fig. 3, where O is the centre of the hexagon and M is the midpoint of a side of the hexagon, show that

$$h^2 = 225 - 30\sqrt{3}x. \qquad [3]$$

(ii) Hence show that the volume V of the pyramid is given by

$$V^2 = 180x^4(15 - 2\sqrt{3}x). \qquad [3]$$

(iii) Use differentiation to find the maximum value of V, proving that it is a maximum. [5]

(iv) Determine the value of h for which V is maximum. [1]

10 The plane p contains the point A with coordinates $(-3,4,-2)$ and the line l with equation $x+2=\frac{4-y}{3}, z=0$.

(i) Find a cartesian equation of p. [3]

(ii) Find a vector equation of the line which is a reflection of l in the y-axis. [4]

The line m passes through A and the point $(-9,9,-6)$.

(iii) Find the acute angle between l and m. [2]

(iv) Find the coordinates of the points on m that are equidistant from p and the x-y plane. [4]

Pioneer Junior College
H2 Mathematics **JC2 2017**
JC2 H2 Preliminary Examination Paper 1 (Solution)

Q1

$$u_n = an^2 + bn + c$$

$$u_1 = a(1)^2 + b(1) + c = 70 \quad \Rightarrow \quad a + b + c = 70 \qquad (1)$$

$$u_2 = a(2)^2 + b(2) + c = 136 \quad \Rightarrow \quad 4a + 2b + c = 136 \qquad (2)$$

$$u_3 = a(3)^2 + b(3) + c = 198 \quad \Rightarrow \quad 9a + 3b + c = 198 \qquad (3)$$

Using GC

$a = -2, \quad b = 72, \quad c = 0$

$u_n = -2n^2 + 72n$

Q2

(i)

$$u_1 = u_0 + 2 - 1 = \frac{3}{2} + 2 - 1 = \frac{5}{2}$$

$$u_2 = u_1 + 2^2 - 2 = \frac{5}{2} + 4 - 2 = \frac{9}{2}$$

$$u_3 = u_2 + 2^3 - 3 = \frac{9}{2} + 8 - 3 = \frac{19}{2}$$

(ii)

Note :

RHS (by sum of first n terms of GP and sum of first n terms of AP)

LHS (by method of difference)

$$u_n - u_{n-1} = 2^n - n$$

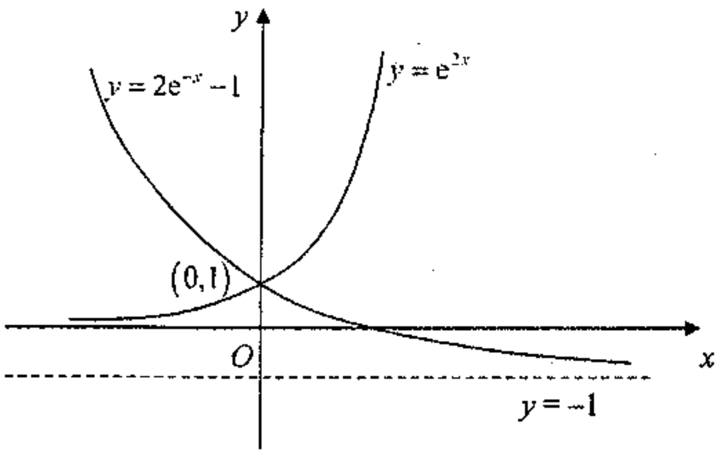
$$\sum_{r=1}^{n} (u_r - u_{r-1}) = \sum_{r=1}^{n} 2^r - r$$
$$= \sum_{r=1}^{n} 2^r - \sum_{r=1}^{n} r$$

$$\begin{aligned}
& u_1 - u_0 \\
& + u_2 - u_1 \\
& + u_3 - u_2 \\
& + \\
& \vdots \\
& + u_{n-2} - u_{n-3} = \frac{2(1-2^n)}{1-2} - \frac{n(n+1)}{2} \\
& + u_{n-1} - u_{n-2} \\
& + u_n - u_{n-1}
\end{aligned}$$

$$u_n - u_0 = \frac{2(1-2^n)}{1-2} - \frac{n(n+1)}{2}$$

$$u_n = -2(1-2^n) - \frac{n(n+1)}{2} + \frac{3}{2}$$
$$= 2^{n+1} - \frac{1}{2} - \frac{n(n+1)}{2}$$

Q3



$e^{2x} \geq 2e^{-x} - 1$
$x \geq 0$

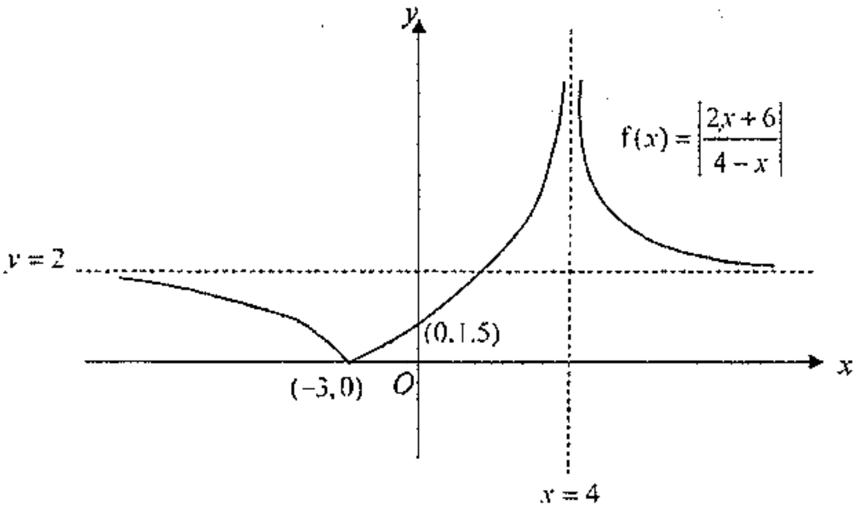
For $x \geq 0$, $e^{2x} \geq 2e^{-x} - 1 \Rightarrow e^{2x} - 2e^{-x} + 1 \geq 0$
For $x < 0$, $e^{2x} - 2e^{-x} + 1 < 0$.

$$\int_{-1}^{2} \left| e^{2x} - 2e^{-x} + 1 \right| dx = \int_{-1}^{0} -(e^{2x} - 2e^{-x} + 1)\, dx + \int_{0}^{2} (e^{2x} - 2e^{-x} + 1)\, dx$$
$$= -\left[\frac{1}{2}e^{2x} + 2e^{-x} + x\right]_{-1}^{0} + \left[\frac{1}{2}e^{2x} + 2e^{-x} + x\right]_{0}^{2}$$
$$= -\left[\left(\frac{1}{2}+2\right) - \left(\frac{1}{2}e^{-2} + 2e - 1\right)\right] + \left[\left(\frac{1}{2}e^{4} + 2e^{-2} + 2\right) - \left(\frac{1}{2}+2\right)\right]$$
$$= \frac{1}{2}e^{4} + 2e + \frac{5}{2}e^{-2} - 4$$

Q4

(i)

$R_f = [0, \infty)$



(ii)

$R_f = [0, \infty)$
$D_f = (-\infty, 4) \cup (4, \infty)$ or $D_f = \mathbb{R} \setminus \{4\}$
$R_f \not\subset D_f$
f^2 does not exists.

(iii)

$k = -3$

(iv)

For $D_f = (-\infty, -3]$

$$y = -\left(\frac{2x+6}{4-x}\right)$$

$$y = \frac{2x+6}{x-4}$$

$$yx - 2x = 6 + 4y$$

$$x = \frac{6+4y}{y-2}$$

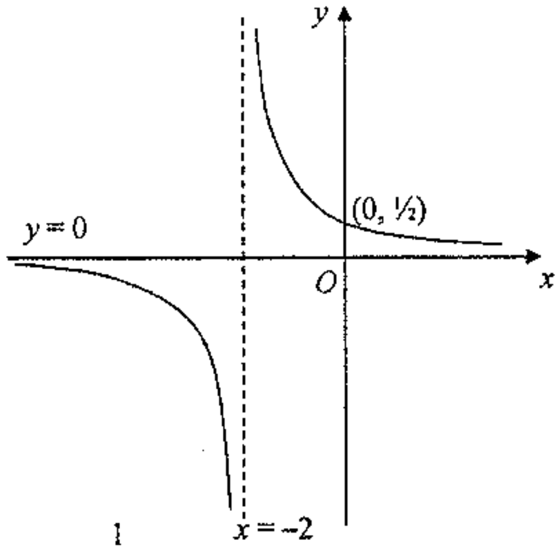
$$f^{-1} : x \mapsto \frac{6+4x}{x-2}, \qquad x \in \mathbb{R},\ 0 \le x < 2$$

Note :

Consider the graph without modulus.

Q5

(i)



(ii)

$x = 2t - 1 \qquad y = \dfrac{1}{2t+1}$

$\dfrac{dx}{dt} = 2 \qquad \dfrac{dy}{dt} = -\dfrac{2}{(2t+1)^2}$

$\dfrac{dy}{dx} = -\dfrac{1}{(2t+1)^2}$

At the point $P(-1, 1)$, $t = 0$

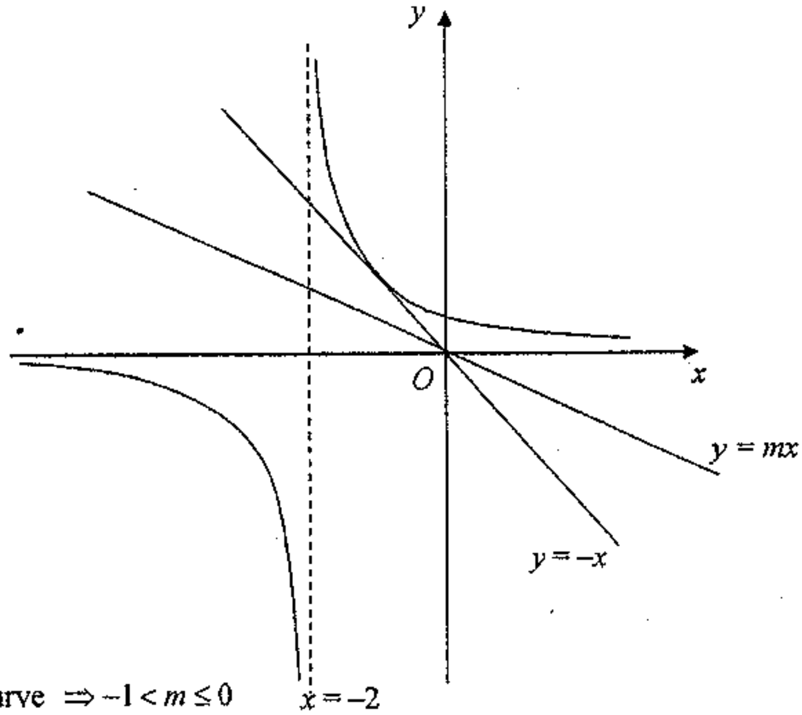
$\dfrac{dy}{dx} = -1$

Equation of tangent at P is

$y - 1 = -1(x+1)$

$y = -x$

(iii)



The line $y = mx$ does not cut the curve $\Rightarrow -1 < m \le 0$

(iv)

Gradient of normal at $P = 1$

Equation of normal at P is

$y - 1 = x - (-1)$

$y = x + 2$

Subst $x = 2t - 1$, $y = \dfrac{1}{2t+1}$ into $y = x + 2$

$\dfrac{1}{2t+1} = 2t - 1 + 2 = 2t + 1$

$(2t+1)^2 = 1$

$2t + 1 = \pm 1$

$t = 0 \quad \text{or} \quad t = -1$

At the point Q, $t = -1$

$x = 2(-1) - 1 = -3$, $y = \dfrac{1}{2(-1)+1} = -1$

Coordinates of Q are $(-3, -1)$

Q6

(i)

AP with $a = 400$, $d = -5$

$S_n = 8500$

$\dfrac{n}{2}\left[2(400) + (n-1)(-5)\right] = 8500$

$5n^2 - 805n + 17000 = 0$

$n = 25$ or $n = 136$ (rejected as already reached peak when $n = 25$)

(ii)

GP with $a = 800$, $r = 90$

$S_{n(AP)} > S_{n(GP)}$

$\dfrac{n}{2}\left[2(400) + (n-1)(-5)\right] > \dfrac{800(1-0.9^n)}{1-0.9}$

$805n - 5n^2 > 16000\left(1 - 0.9^n\right)$

Using GC,

$n \geq 20$

A will overtake B on the 20th day.

NORMAL FLOAT AUTO REAL RADIAN MP
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X	Y1	Y2
18	6435	6799.2
19	6745	6919.3
20	7050	7027.4
21	7350	7124.6
22	7645	7212.2
23	7935	7291
24	8220	7361.9
25	8500	7425.7
26	8775	7483.1
27	9045	7534.8
28	9310	7581.3

X=18

(iii)

$S_\infty = \dfrac{800}{1-0.9} = 8000 (< 8500)$

Hence, Team B will never be able to reach the peak.

(iv)

$T_{15} = 800\left(0.9^{15-1}\right) = 183.014$

$S_{15} = \dfrac{800\left(1-0.9^{15}\right)}{1-0.9} = 6352.871$

Remaining distance $= 8500 - 6352.871 = 2147.129$

First term of new GP $= 183.014 \times 0.95 = 173.864$

$S_{n(New\,GP)} = 2147.129$

$\dfrac{173.864\left(1-0.95^{n}\right)}{1-0.95} = 2147.129$

$0.95^{n} = 0.38253$

$n = 18.7$

Team B will take $15+19=34$ days

Hence, Team A will reach the peak first.

Q7

(i)

Consider the graph of $y = 2 + \dfrac{x-3}{(x-2)(x+1)}$ and $y = p$ intersecting.

$p = 2 + \dfrac{x-3}{(x-2)(x+1)}$

$p - 2 = \dfrac{x-3}{x^2-x-2}$

$px^2 - px - 2p - 2x^2 + 2x + 4 = x - 3$

$(p-2)x^2 + (1-p)x + (7-2p) = 0$

Discriminant ≥ 0

$(1-p)^2 - 4(p-2)(7-2p) \geq 0$

$1 - 2p + p^2 - 28p + 8p^2 + 56 - 16p \geq 0$

$9p^2 - 46p + 57 \geq 0$

$(9p-19)(p-3) \geq 0$

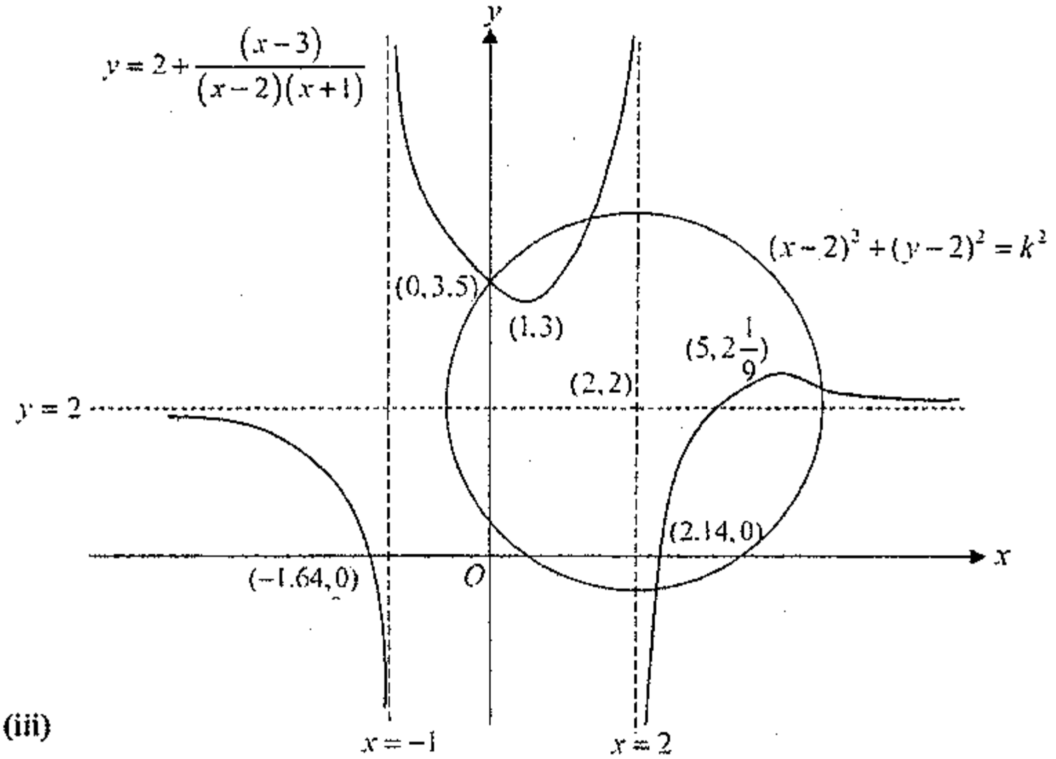
$p \leq \dfrac{19}{9}$ or $p \geq 3$

$y \leq 2\dfrac{1}{9}$ or $y \geq 3$

> Note :
>
> Finding y values by $\dfrac{dy}{dx} = 0$ is not encouraged.

$+ \quad \frac{19}{9} \quad - \quad 3 \quad +$

(ii)



(iii)

$$(x-2)^2 + \frac{(x-3)^2}{(x-2)^2(x+1)^2} = k^2$$

$$(x-2)^2 + (y-2)^2 = k^2$$

Distance from centre of circle to the $y-$intercept of $y = 2 + \dfrac{(x-3)}{(x-2)(x+1)}$

$$= \sqrt{2^2 + \left(\frac{7}{2} - 2\right)^2} = \frac{5}{2}$$

$k < -2.5$ or $k > 2.5$

> Note :
>
> 1) Be mindful of the link between (i) and (ii). The values 3 and $\frac{19}{9}$ must be some special points on the graph. Look out for those points.

Q8

(a)

$$\int \sqrt{\frac{1-x}{x}}\,\mathrm{d}x$$
$$= \int \sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}\,2\sin\theta\cos\theta\,\mathrm{d}\theta$$
$$= \int 2\cos^2\theta\,\mathrm{d}\theta$$
$$= \int (1+\cos 2\theta)\,\mathrm{d}\theta$$
$$= \theta + \frac{1}{2}\sin 2\theta + C$$
$$= \theta + \sin\theta\cos\theta + C$$
$$= \sin^{-1}\left(\sqrt{x}\right) + \sqrt{x\left(1-x\right)} + C$$

$x = \sin^2\theta$

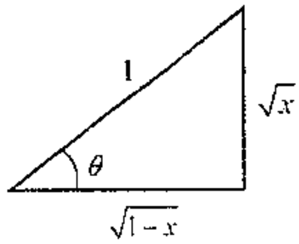
$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sin\theta\cos\theta$

Since $\sqrt{x} = \sin\theta$

Consider a right angle triangle

or use trigo identity

$\cos^2\theta + \sin^2\theta = 1$

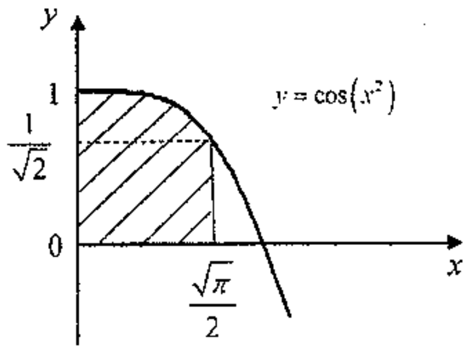


(b)

$y = \cos\left(x^2\right)$

$x = 0 \Rightarrow y = 1$

$x = \dfrac{\sqrt{\pi}}{2} \Rightarrow y = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$



Required volume

$$= \pi\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right) + \pi\int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y\,\mathrm{d}y$$
$$= \frac{\pi^2}{4\sqrt{2}} + \pi\int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y\,\mathrm{d}y$$

$$\int \cos^{-1} y\,\mathrm{d}y$$

Let $u = \cos^{-1} y$ $\qquad \dfrac{\mathrm{d}v}{\mathrm{d}y} = 1$

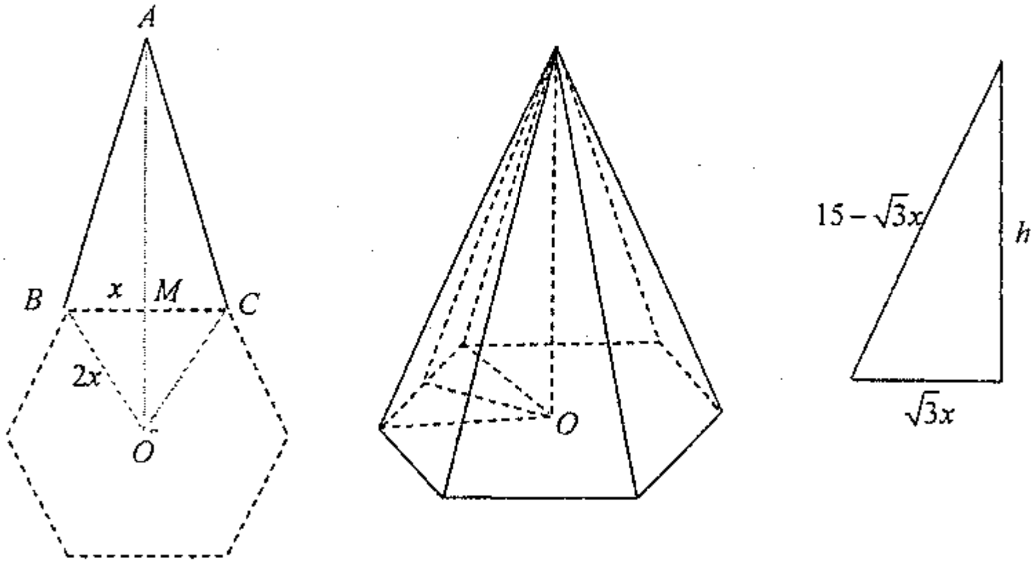
$\dfrac{\mathrm{d}u}{\mathrm{d}y} = -\dfrac{1}{\sqrt{1-y^2}}$ $\qquad v = y$

$$= y\cos^{-1} y - \int -\frac{y}{\sqrt{1-y^2}}\,\mathrm{d}y$$
$$= y\cos^{-1} y - \frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}}\,\mathrm{d}y$$
$$= y\cos^{-1} y - \frac{1}{2}[2(1-y^2)^{\frac{1}{2}}] + c$$
$$= y\cos^{-1} y - \sqrt{1-y^2} + c$$

Required volume $= \dfrac{\pi^2}{4\sqrt{2}} + \pi\left[y\cos^{-1} y - \sqrt{1-y^2}\right]_{\frac{1}{\sqrt{2}}}^{1}$

$$= \frac{\pi^2}{4\sqrt{2}} + \pi\left[0 - \left(\frac{1}{\sqrt{2}}\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)\right]$$
$$= \frac{\pi}{\sqrt{2}}$$

Q9



$OM = \sqrt{(2x)^2 - x^2} = \sqrt{3}x$

$\therefore \quad AM = 15 - \sqrt{3}x$

Let h cm be the height of the pyramid.

$$\begin{aligned} h^2 &= (15 - \sqrt{3}x)^2 - (\sqrt{3}x)^2 \\ &= 225 - 30\sqrt{3}x + 3x^2 - 3x^2 \\ &= 225 - 30\sqrt{3}x \qquad \text{(shown)} \end{aligned}$$

$$\begin{aligned} \text{Area of hexagon} &= 6 \times \text{ area of triangle } OBC \\ &= 6(\tfrac{1}{2})(2x)(\sqrt{3}x) \\ &= 6\sqrt{3}x^2 \end{aligned}$$

$\therefore \quad V = \frac{1}{3}(6\sqrt{3}x^2)\sqrt{225 - 30\sqrt{3}x}$

$V^2 = 180x^4(15 - 2\sqrt{3}x)$ (shown)

$V^2 = 180(15x^4 - 2\sqrt{3}x^5)$

Differentiating wrt x,

$$\begin{aligned} 2V\frac{dV}{dx} &= 180\left(60x^3 - 10\sqrt{3}x^4\right) \\ &= 1800x^3\left(6 - \sqrt{3}x\right) \end{aligned}$$

$\frac{dV}{dx} = 0 \Rightarrow x = 0 \quad$ or $\quad x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

(NA as $x > 0$)

Alternatively,

$V = 6\sqrt{5}x^2)\sqrt{15 - 2\sqrt{3}x}$

$$\begin{aligned} \frac{dV}{dx} &= (6\sqrt{5}x^2)\frac{1}{2}(15 - 2\sqrt{3}x)^{-\frac{1}{2}}(-2\sqrt{3}) \\ &\quad + (15 - 2\sqrt{3}x)^{\frac{1}{2}}(12\sqrt{5}x) \\ &= 12\sqrt{5}x(15 - 2\sqrt{3}x)^{\frac{1}{2}} - 6\sqrt{15}x^2(15 - 2\sqrt{3}x)^{-\frac{1}{2}} \\ &= 6\sqrt{5}x(15 - 2\sqrt{3}x)^{-\frac{1}{2}}[2(15 - 2\sqrt{3}x) - \sqrt{3}x] \\ &= \frac{30\sqrt{5}x(6 - \sqrt{3}x)}{\sqrt{(15 - 2\sqrt{3}x)}} \end{aligned}$$

$\frac{dV}{dx} = 0 \Rightarrow x = 0 \quad$ or $\quad x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

(NA as $x > 0$)

To Prove Maximum

Note :

You need to be clear with the values for the test and conclude < or > 0.

Method 1

$$2V\frac{d^2V}{dx^2}+2\left(\frac{dV}{dx}\right)^2=180\left[180x^2-40\sqrt{3}x^3\right]$$

$$x=2\sqrt{3},\ \frac{d^2V}{dx^2}=\frac{180}{2V}\left[180\left(2\sqrt{3}\right)^2-40\sqrt{3}\left(2\sqrt{3}\right)^3\right]=-\frac{64800}{V}<0 \text{ since } V>0$$

Method 2

x	3.4	$2\sqrt{3}=3.46$	3.5
$\frac{dV}{dx}$	$\approx\frac{7855}{2V}>0$	0	$\approx-\frac{4799}{2V}<0$

V is maximum when $x=2\sqrt{3}$ cm.
Max $V=72\sqrt{15}$ cm^3.

(iv)
When $x=2\sqrt{3},\ h^2=225-30\sqrt{3}(2\sqrt{3})=45$
$h=3\sqrt{5}$ cm (reject $h=-3\sqrt{5}$ as $h>0$)

Q10
(i)

$l: x+2=\frac{4-y}{3}, z=0$

$$l: \mathbf{r}=\begin{pmatrix}-2\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\0\end{pmatrix} \qquad \lambda\in\mathbb{R}$$

$$\begin{pmatrix}-2\\4\\0\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\-3\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-6\\-2\\3\end{pmatrix}$$

$$r\bullet\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}\bullet\begin{pmatrix}6\\2\\-3\end{pmatrix}=-4$$

$p: 6x+2y-3z=-4$

(ii)

To find intersection between y-axis and l, sub $x=0$ into l

$$0+2=\frac{4-y}{3}\Rightarrow y=-2$$

Thus, point of intersection is $(0,-2,0)$.

Point of reflection of $(-2,4,0)$ about y-axis is $(2,4,0)$

$$\begin{pmatrix}2\\4\\0\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\6\\0\end{pmatrix}=2\begin{pmatrix}1\\3\\0\end{pmatrix}$$

Line of reflection, $l':\mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+s\begin{pmatrix}1\\3\\0\end{pmatrix} \qquad s\in\mathbb{R}$

(iii)

$$\begin{pmatrix}-9\\9\\-6\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}=\begin{pmatrix}-6\\5\\-4\end{pmatrix}$$

$$\left|\begin{pmatrix}-6\\5\\-4\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\0\end{pmatrix}\right|=\left|\begin{pmatrix}-6\\5\\-4\end{pmatrix}\right|\left|\begin{pmatrix}1\\-3\\0\end{pmatrix}\right|\cos\theta$$

$$\cos\theta=\frac{21}{\sqrt{(-6)^2+5^2+(-4)^2}\sqrt{1^2+3^2}}=\frac{21}{\sqrt{770}}$$

$$\theta=40.8^\circ$$

(iv)

Let the point that is equidistant from both planes be C.

$\overrightarrow{OC}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}+t\begin{pmatrix}6\\-5\\4\end{pmatrix}$ for some $t\in\mathbb{R}$

Distance of C from p = Distance of C from x-y plane

$$\frac{\left|\left[\begin{pmatrix}-3+6t\\4-5t\\-2+4t\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}\right|}{\sqrt{6^2+2^2+3^2}}=\frac{\left|\left[\begin{pmatrix}-3+6t\\4-5t\\-2+4t\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{0^2+0^2+1^2}}$$

$$\frac{|36t-10t-12t|}{7}=|-2+4t|$$

$$|t|=|2t-1|$$

$t^2=4t^2-4t+1$	$t=2t-1$ or $t=-2t+1$
$3t^2-4t+1=0$	

$t=1$ or $t=\frac{1}{3}$

$\overrightarrow{OC}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}+(1)\begin{pmatrix}6\\-5\\4\end{pmatrix}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$ or $\overrightarrow{OC}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}+\left(\frac{1}{3}\right)\begin{pmatrix}6\\-5\\4\end{pmatrix}=\left(\frac{1}{3}\right)\begin{pmatrix}-3\\7\\-2\end{pmatrix}$

The 2 points are $(3,-1,2)$ and $\left(-1,\frac{7}{3},-\frac{2}{3}\right)$.

Candidate Name: ______________________ Class: __________

PIONEER JUNIOR COLLEGE
While I Live, I Learn

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS **9758/02**
Paper 2
18 Sept 2017
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 Referred to origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P lies on OA produced such that $OA : AP = 1 : \lambda$. Point Q lies on OB, between O and B, such that $OQ : QB = 3 : 1$. The mid-point of PB is M. Show that the ratio of the area of triangle OPM to the area of triangle OQM is independent of λ. [5]

2 By differentiating $\cos x \dfrac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x, solve the differential equation

$$\cos x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \sin x \frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x + \cos 2x ,$$

giving y in terms of x. [6]

3 **(a)** State a sequence of transformations which transform the graph of $y = \ln(2x+1)$ to the graph of $y = \ln\left(\dfrac{3}{2x-1}\right)$. [3]

(b) It is given that

$$f(x) = \begin{cases} ax & 0 \le x < 1, \\ a & 1 \le x \le 2, \\ 3a - ax & 2 < x \le 3, \end{cases}$$

and that $f(x+3) = \dfrac{1}{2} f(x)$, for all real values of x, where a is a positive constant.

(i) Sketch the graph of $y = f(x)$ for $-2 \le x \le 8$. [3]

(ii) Find, in terms of a, $\int_0^2 f(-x)\,\mathrm{d}x$. [1]

(iii) Find the value of the constant a for which $\int_0^\infty f(x)\,\mathrm{d}x = 16$. [2]

4 **Do not use a graphic calculator in answering this question.**

The complex number z is given by $z=-1+\mathrm{i}c$, where c is a non-zero real number. Given that $\frac{z^n}{z^*}$ is purely real, find

(i) the possible values of c when $n=2$, [4]

(ii) the three smallest positive integer values of n when $c=\sqrt{3}$. [5]

5 It is given that $y=\sec 2x$.

(i) Show that $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y^2(y^2-1)$. [3]

(ii) By further differentiation, find the Maclaurin's series for y up to and including the term in x^4. [5]

(iii) By considering $\sec 2x=\frac{1}{\cos 2x}$, check on the correctness of your answer in part **(ii)**. [3]

Section B: Statistics [60 marks]

6 An unbiased disc has a single dot marked on one side and two dots marked on the other side. A tetrahedral die has faces marked with score of 1, 2, 3, and 4. The probability of getting a score of 1, 2, 3, and 4 is $\frac{1}{5}, p, \frac{1}{5}$ and q respectively, where $p,q\in[0,1]$.

A game is played by throwing the disc and the die together. The random variable S is the sum of the score showing on the die and twice the number of dots showing on the disc.

(i) Find $\mathrm{P}(S=6)$. [2]

Given that $\mathrm{P}(S=4)=\frac{1}{6}$,

(ii) calculate the values of p and q, [2]

(iii) and find the probability distribution of S. [2]

7 The masses, in kilograms, of black sea bass fish and red tilapia fish sold in a supermarket are normally distributed. The means and standard deviations of these distributions, and the selling prices, in $ per kilogram, are given in the following table.

	Mean Mass (kg)	Standard Deviation (kg)	Selling Price ($ per kg)
Black sea bass fish	1.10	0.20	12
Red tilapia fish	0.55	0.05	9

(i) Ayden bought 2 black sea bass fish and 3 red tilapia fish. Find the probability that he pays more than $40. State an assumption needed in your calculation. [4]

(ii) Five red tilapia fish are randomly chosen. Find the probability that the fifth red tilapia fish is the third red tilapia fish weighing less than half a kilogram. [3]

8 The average amount of cholesterol in one standard fillet of raw red snapper from a fish farm is w mg. To lower the cost of operations, the farmer decides to use a cheaper mixture of fish feed. The farmer conducts a test to check if the average amount of cholesterol in one standard fillet of raw red snapper is affected by the change of fish feed. 50 standard fillets of raw red snapper from 50 different fish were taken and the average amount of cholesterol in these fillets is found to be 78.5 mg, with a standard deviation of 2 mg.

(i) Given that at 5% level of significance, there is insufficient evidence to conclude that the mean amount of cholesterol in one standard fillet of raw red snapper is affected, find the range of possible values of w. [5]

(ii) State, giving a reason, whether any assumptions about the population are needed in order for the test to be valid. [2]

9 A student working on a coding project studies 11-digit quaternary sequences. A quaternary sequence is a sequence formed using the digits 0, 1, 2 or 3. Examples of such sequences are 12030201131, 01122211100, 12321232123 and 00000000000. Find the number of ways that 11-digit quaternary sequences can be formed with

(i) no restriction, [1]

(ii) exactly four 0s and four 2s, [3]

(iii) at least two consecutive digits that are the same. [3]

10 For events A and B, it is given that $P(A)=\frac{11}{20}$ and $P(B)=\frac{1}{2}$.

(i) Find the greatest and least possible values of $P(A\cap B)$. [2]

It is given in addition that $P(B\mid A')=\frac{7}{9}$.

(ii) Find $P(A\cup B)$. [2]

(iii) Determine if A and B are independent events. Justify your answer. [2]

(iv) Given another event C such that $P(C)=\frac{2}{5}$, $P(A\cup B\cup C)=\frac{19}{20}$, $P(A\cap B\cap C)=\frac{1}{10}$ and $P(A\cap C)=2P(B\cap C)$, find $P(A\cap C)$. [3]

11 Based on past statistical data, there is a 7% chance that a passenger with reservation for a flight will not show up. In order to maximise revenue, airline companies accept more reservations than the passenger capacity of its planes. State 2 assumptions needed such that the number of passengers who do not show up for a flight may be well modelled by a Binomial distribution. [2]

An airline company operates a flight from Singapore to Maluku on Boeing 737-200 planes, which has capacity of 232 passengers each.

(i) Find the probability that when 245 reservations are accepted, the flight is overbooked, i.e. there is not enough seats available for the passengers who show up. [2]

(ii) Find the maximum number of reservations that should be accepted in order to ensure that the probability of overbooking is less than 1%. [3]

This flight operates once daily throughout the year and 245 reservations are accepted for each flight.

(iii) Find the probability that no flight is overbooked in a week. [2]

(iv) Taking a year as 52 weeks, estimate the probability that the mean number of flights that is overbooked in each week for the year is not more than 1. [3]

12 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each case your diagram should include 5 points, approximately equally spaced with respect to x, and with all x- and y- values positive. The letters p, q, r, s, t and u represent constants.

(A) $y = p + qx^2$, where p is positive and q is negative,

(B) $y = r + se^x$, where r is negative and s is positive,

(C) $y = t + \dfrac{u}{x}$, where t is positive and u is positive. [3]

Daisy enrolled in a weight management programme to reduce her weight. Her weight, y kg at the end of week x of the programme are given in the table.

x	1	2	3	4	5	6	7
y	74.9	72.9	71.6	70.8	70.4	70.2	70.1

(ii) Draw a scatter diagram to illustrate the data. [2]

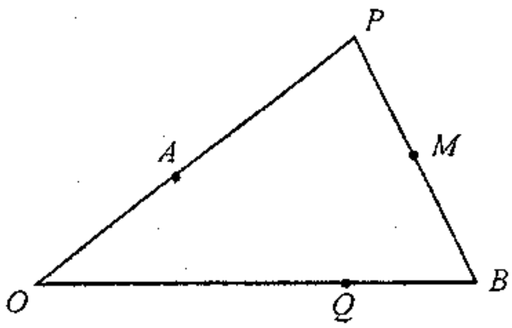
(iii) Explain which of the three cases in part **(i)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

(iv) Use the case that you identified in part **(iii)** to find the equation of a suitable regression line, and use your equation to estimate Daisy's weight at the end of week 10. [3]

(v) Given that 1 week = 7 days, re-write your equation from part **(iv)** so that it can be used to estimate the weight when the time period of the programme is given in days. [2]

Pioneer Junior College
H2 Mathematics **JC2 2017**
JC2 H2 Preliminary Examination Paper 2 (Solution)

Q1



$\overrightarrow{OP}=(\lambda+1)\mathbf{a}$

$$\overrightarrow{OM}=\frac{\overrightarrow{OP}+\overrightarrow{OB}}{2}=\frac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}$$

$$\text{area of triangle } OPM=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OM}\right|=\frac{1}{2}\left|(\lambda+1)\mathbf{a}\times\frac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}\right|=\frac{(\lambda+1)}{4}|\mathbf{a}\times\mathbf{b}|$$

$$\text{area of triangle } OQM=\frac{1}{2}\left|\overrightarrow{OQ}\times\overrightarrow{OM}\right|=\frac{1}{2}\left|\frac{3}{4}\mathbf{b}\times\frac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}\right|=\frac{3(\lambda+1)}{16}|\mathbf{a}\times\mathbf{b}|$$

Ratio of the area of triangle OPM to the area of triangle OQM is

$\frac{(\lambda+1)}{4}:\frac{3(\lambda+1)}{16}=4:3$ (Shown)

Q2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right]=\cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\sin x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right]=\sec^2 x+\cos 2x$$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=\int \sec^2 x+\cos 2x\,\mathrm{d}x$$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=\tan x+\frac{1}{2}\sin 2x+C$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sec x\tan x+\sin x+C\sec x$$

$$y=\int \sec x\tan x+\sin x+C\sec x\,\mathrm{d}x$$

$$y=\sec x-\cos x+C\ln\left|\sec x+\tan x\right|+D$$

Q3

(a)

Method 1

Step 1 : Translate by 1 unit in the direction of the x-axis.

Step 2 : reflection about the x-axis.

Step 3 : Translate by ln3 units in the direction of the y-axis.

$$y=\ln(2x+1)\to y=\ln\left[2(x-1)+1\right]\to y=-\ln(2x-1)\to y=\ln 3-\ln(2x-1)=\ln\left(\frac{3}{2x-1}\right)$$

Method 2

Step 1 : reflection about the x-axis.

Step 2 : Translate by 1 unit in the direction of the x-axis.

Step 3 : Translate by ln3 units in the direction of the y-axis.

$$y=\ln(2x+1)\to y=-\ln(2x+1)\to y=-\ln\left[2(x-1)+1\right]\to y=\ln 3-\ln(2x-1)=\ln\left(\frac{3}{2x-1}\right)$$

Method 3

Step 1 : reflection about the x-axis.

Step 2 : Translate by ln3 units in the direction of the y-axis.

Step 3 : Translate by 1 unit in the direction of the x-axis.

$$y=\ln(2x+1)\to y=-\ln(2x+1)\to y=\ln 3-\ln(2x+1)\to y=\ln 3-\ln\left[2(x-1)+1\right]=\ln\left(\frac{3}{2x-1}\right)$$

Method 4

Step 1 : Translate by 1 unit in the direction of the x-axis.

Step 2 : Translate by $-\ln 3$ units in the direction of the y-axis.

Step 3 : reflection about the x-axis.

$$y=\ln(2x+1)\to y=\ln\left[2(x-1)+1\right]\to y=-\ln 3+\ln\left(2x-1\right)\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right)$$

Method 5

Step 1 : Translate by $-\ln 3$ units in the direction of the y-axis.

Step 2 : Translate by 1 unit in the direction of the x-axis.

Step 3 : reflection about the x-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln 3+\ln(2x+1)\to y=-\ln 3+\ln\left[2\left(x-1\right)+1\right]\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right)$$

Method 6

Step 1 : Translate by $-\ln 3$ units in the direction of the y-axis.

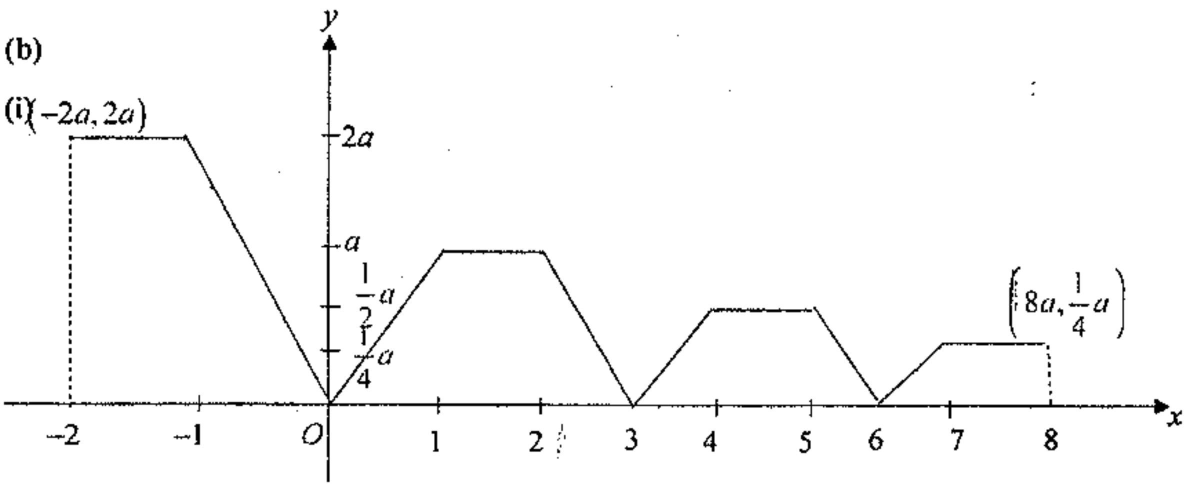
Step 2 : reflection about the x-axis.

Step 3 : Translate by 1 unit in the direction of the x-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln 3+\ln(2x+1)\to y=\ln 3-\ln(2x+1)\to y=\ln 3-\ln\left[2\left(x-1\right)+1\right]=\ln\left(\frac{3}{2x-1}\right)$$

(b)

(i)



(ii)

$$\int_0^2 f(-x)\,dx = \frac{1}{2}(2+1)(2a) = 3a$$

(iii)

$$\int_0^\infty f(x)\,dx = 16$$

$$2\left(a+\frac{1}{2}a+\frac{1}{4}a+......\right)=16$$

$$2a\left(\frac{1}{1-\frac{1}{2}}\right)=16$$

$$4a=16$$

$$a=4$$

Note

(ii) :

$$\int_0^2 f(-x)\,dx = \int_{-2}^0 f(x)\,dx$$

which is the area of the trapezium from $x=-2$ to $x=0$

(iii)

Consider the graph for $x > 0$, which is equivalent to infinite number of rectangles, each of length 2, and breadth $a, \frac{1}{2}a, \frac{1}{4}a....$ which follows a GP.

Q4

(i)

$$\frac{z^2}{z^*}=\frac{(-1+ic)^2}{(-1-ic)}$$

$$=\frac{1-i2c-c^2}{(-1-ic)}$$

$$=\frac{1-i2c-c^2}{(-1-ic)}\times\frac{(-1+ic)}{(-1+ic)}$$

$$=\frac{-1+ic+i2c+2c^2+c^2-ic^3}{1+c^2}$$

Since $\frac{z^2}{z^*}$ is purely real,

$$\frac{3c-c^3}{1+c^2}=0$$

$$c(3-c^2)=0$$

$c=0$ (rej since c is non-zero) $c=\pm\sqrt{3}$

(ii)

$$z=-1+i\sqrt{3}$$

$|z|=2,\qquad \arg(z)=\frac{2\pi}{3}$

$$\frac{z^n}{z^*}=\frac{\left(2e^{i\frac{2\pi}{3}}\right)^n}{2e^{i\left(-\frac{2\pi}{3}\right)}}=2^{n-1}e^{i\left(\frac{2n\pi}{3}+\frac{2\pi}{3}\right)}$$

Since $\frac{z^n}{z^*}$ is purely real,

$\arg\left(\frac{z^n}{z^*}\right)=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,\pm6\pi...$ or $\qquad \sin\left[(n+1)\frac{2\pi}{3}\right]=0$

$(n+1)\frac{2\pi}{3}=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,\pm6\pi,..$

$n+1=0,\pm\frac{3}{2},\pm3,\pm\frac{9}{2},\pm6,...$

Considering positive integer values only,

$n+1=3,6,9...$

Three smallest positive integer values of n are $2,5,8$

Q5

(i)

$y=\sec 2x$

$\frac{dy}{dx}=2\sec 2x\tan 2x=2y\tan 2x$

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2&=4y^2\tan^2 2x\\&=4y^2(\sec^2 2x-1)\\&=4y^2(y^2-1)\end{aligned}$$

(ii)

$2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=16y^3\frac{dy}{dx}-8y\frac{dy}{dx}$

$\frac{d^2y}{dx^2}=8y^3-4y$

$\frac{d^3y}{dx^3}=24y^2\frac{dy}{dx}-4\frac{dy}{dx}$

$\frac{d^4y}{dx^4}=24y^2\frac{d^2y}{dx^2}+48y\left(\frac{dy}{dx}\right)^2-4\frac{d^2y}{dx^2}$

When $x=0$, $y=1$, $\frac{dy}{dx}=0$, $\qquad \frac{d^2y}{dx^2}=4$, $\qquad \frac{d^3y}{dx^3}=0$, $\qquad \frac{d^4y}{dx^4}=80$

$y = 1 + \frac{x^2}{2!}(4) + \frac{x^4}{4!}(80) + \ldots$

$y = 1 + 2x^2 + \frac{10}{3}x^4 + \ldots$

(iii)

$\sec 2x = \frac{1}{\cos 2x}$

$\approx \frac{1}{1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}}$

$= (1 - 2x^2 + \frac{2}{3}x^4)^{-1}$

$= 1 + (-1)(-2x^2 + \frac{2}{3}x^4) + \frac{(-1)(-2)}{2!}\left(-2x^2 + \frac{2}{3}x^4\right)^2 + \ldots$

$= 1 + 2x^2 - \frac{2}{3}x^4 + 4x^4 + \ldots$

$\approx 1 + 2x^2 + \frac{10}{3}x^4$

Q6

(i)

		Score on the die			
Dots on the disc	S	**1** $(\frac{1}{5})$	**2** (p)	**3** $(\frac{1}{5})$	**4** (q)
	1 $(\frac{1}{2})$	3 $(\frac{1}{10})$	4 $(\frac{1}{2}p)$	5 $(\frac{1}{10})$	6 $(\frac{1}{2}q)$
	2 $(\frac{1}{2})$	5 $(\frac{1}{10})$	6 $(\frac{1}{2}p)$	7 $(\frac{1}{10})$	8 $(\frac{1}{2}q)$

Since total probability = 1, $\frac{1}{5} + p + \frac{1}{5} + q = 1 \Rightarrow p + q = \frac{3}{5}$

OR: $\left(\frac{1}{10}\right) + \left(\frac{1}{2}p\right) + \left(\frac{1}{5}\right) + \left(\frac{1}{2}p + \frac{1}{2}q\right) + \left(\frac{1}{10}\right) + \left(\frac{1}{2}q\right) = 1 \Rightarrow p + q = \frac{3}{5}$

Hence $\mathrm{P}(S = 6) = \frac{1}{2}p + \frac{1}{2}q = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$.

(ii)

$\mathrm{P}(S = 4) = \frac{1}{6} \Rightarrow \frac{1}{2}p = \frac{1}{6} \Rightarrow p = \frac{1}{3}$

Since $p + q = \frac{3}{5}$, then $q = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}$.

(iii)

s	3	4	5	6	7	8
$\mathrm{P}(S = s)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{15}$

Q7

(i)

$X \sim$ mass of a black sea bass fish. $\quad X \sim \mathrm{N}\left(1.1, 0.2^2\right)$

$Y \sim$ mass of a red tilapia fish. $\quad Y \sim \mathrm{N}\left(0.55, 0.05^2\right)$

Let T be the total cost of 2 black sea bass and 3 red tilapia. Then

$T = 12\left(X_1 + X_2\right) + 9\left(Y_1 + Y_2 + Y_3\right)$

$$\begin{aligned}\mathrm{E}(T) &= (12)(2)\mathrm{E}(X) + (9)(3)\mathrm{E}(Y) \\ &= 26.4 + 14.85 \\ &= 41.25\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= (12)^2(2)\mathrm{Var}(X) + (9)^2(3)\mathrm{Var}(Y) \\ &= 11.52 + 0.6075 \\ &= 12.1275\end{aligned}$$

Thus $T \sim \mathrm{N}\left(41.25\ ,\ 12.1275\right)$.

$\mathrm{P}(T > 40) = 0.64018 \approx 0.640$ (3 s.f.)

An assumption needed is the price / mass of all fish are independent of one another.

(ii)

Probability required $= \dfrac{4!}{2!2!}\,[\mathrm{P}\,(Y > 0.5)]^2\ [\mathrm{P}\,(Y < 0.5)]^3 \approx 0.0170$

Q8

(i)

$X \sim$ amount of cholesterol in one standard fillet of raw red snapper.

unbiased estimate of population variance σ^2 is $s^2 = \dfrac{n}{n-1}\sigma_x{}^2 = \dfrac{50}{49}(2)^2 = \dfrac{200}{49}$

Test $\quad \mathrm{H}_0 : \mu = w \qquad$ vs $\qquad \mathrm{H}_1 : \mu \neq w$

Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(w, \dfrac{\frac{200}{49}}{50}\right)$ approx.

$\overline{X} \sim N\left(w, \dfrac{4}{49}\right)$ approx.

Level of significance: 5 %

Critical region is $z < -1.9600$ or $z > 1.9600$

Standardised test statistic: $z = \dfrac{78.5 - w}{\frac{2}{7}}$

Since H_0 is not rejected, z lies outside the critical region.

$$-1.9600 < \frac{78.5 - w}{\frac{2}{7}} < 1.9600$$

$$-1.96\left(\frac{2}{7}\right) < 78.5 - w < 1.96\left(\frac{2}{7}\right)$$
$$-1.96\left(\frac{2}{7}\right) - 78.5 < -w < 1.96\left(\frac{2}{7}\right) - 78.5$$
$$-79.06 < -w < -77.94$$
$$77.94 < w < 79.06$$

(ii)
No. It is not necessary to assume that amount of cholesterol in a standard fillet follows a normal distribution since sample size is large, by Central Limit Theorem, sample mean is normally distributed approximately.

Q9

(i)
No. of ways = $4^{11} = 4194304$

(ii)
Case 1: four 0s, four 2s, with one 1, two 3s or one 3, two 1s
No. of ways = $\dfrac{11!}{4!4!2!} \times 2 = 69300$
Case 2: four 0s, four 2s, with three 1s or three 3s
No. of ways = $\dfrac{11!}{4!4!3!} \times 2 = 23100$
Hence the total number of ways is
$69300 + 23100 = 92400$

Alternative

No. of ways = ${}^{11}C_4 \times {}^{7}C_4 \times 2^3 = 92400$
Choose 4 slots from 11 slots to place four 0s. Choose 4 slots from remaining 7 slots to place four 2s. 2 choices (digit 1 or 3) for each of the remaining 3 slots.

(iii)
No. of ways = no. of ways without restriction − no. of ways with no consecutive digits that are the same
$= 4194304 - 4 \times 3^{10} = 3958108$

Q10
(i)
Least value of $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - 1 = \dfrac{11}{20} + \dfrac{1}{2} - 1 = \dfrac{1}{20}$

Greatest value of $\mathrm{P}(A \cap B) = \mathrm{P}(B) = \dfrac{1}{2}$

(ii)
$$\mathrm{P}(B \cap A') = \mathrm{P}(B \mid A')\mathrm{P}(A') = \frac{7}{9} \times \frac{9}{20} = \frac{7}{20}$$
$$\mathrm{P}(A \cup B) = \mathrm{P}(B \cap A') + \mathrm{P}(A) = \frac{7}{20} + \frac{11}{20} = \frac{9}{10}$$

(iii)

$\because \mathrm{P}(B \mid A') = \frac{7}{9} \neq \frac{1}{2} = \mathrm{P}(B)$, then B and A' are not independent events.

Hence A and B are not independent events.

OR:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$\frac{9}{10} = \frac{11}{20} + \frac{1}{2} - \mathrm{P}(A \cap B)$$

$$\mathrm{P}(A \cap B) = \frac{3}{20}$$

$$\mathrm{P}(A)\mathrm{P}(B) = \frac{11}{20} \times \frac{1}{2} = \frac{11}{40}$$

$$\mathrm{P}(A \cap B) \neq \mathrm{P}(A)\mathrm{P}(B)$$

Hence, A and B are not independent events.

(iv)

$$\mathrm{P}\left[C \cap (A \cup B)'\right] = \mathrm{P}(A \cup B \cup C) - \mathrm{P}(A \cup B) = \frac{19}{20} - \frac{9}{10} = \frac{1}{20}$$

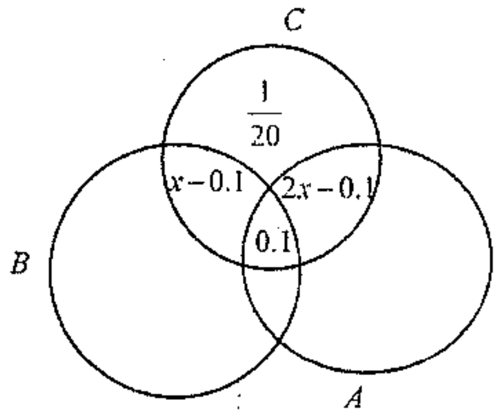
Let $\mathrm{P}(B \cap C) = x$, then $\mathrm{P}(A \cap C) = 2x$

$$\mathrm{P}(C) = \frac{1}{20} + \frac{1}{10} + \left(x - \frac{1}{10}\right) + \left(2x - \frac{1}{10}\right)$$

$$\frac{2}{5} = 3x - \frac{1}{20}$$

$$x = 0.15$$

$$\mathrm{P}(A \cap C) = 2x = 0.3$$



Or

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$$

$$\frac{19}{20} = \frac{11}{20} + \frac{1}{2} + \frac{2}{5} - \frac{3}{20} - 2\mathrm{P}(B \cap C) - \mathrm{P}(B \cap C) + \frac{1}{10}$$

$$3\mathrm{P}(B \cap C) = \frac{9}{20}$$

$$\mathrm{P}(B \cap C) = \frac{3}{20}$$

$$\mathrm{P}(A \cap C) = \frac{6}{20} = \frac{3}{10}$$

Q11

2 assumptions:
- Occurrence of show / no show is independent among passengers.
- Probability that a passenger does not show up is constant.

(i)
$X \sim$ number of passengers with reservation, who show up, out of 245.
$X \sim \mathrm{B}(245\ ,\ 0.93)$
$\mathrm{P}(X > 232) = 1 - \mathrm{P}(X \le 232) = 0.118761 \approx 0.119$ (3 s.f.)

(ii)
$W \sim$ number of passengers with reservation, who show up, out of n.
$W \sim \mathrm{B}(n\ ,\ 0.93)$
$\mathrm{P}(W > 232) < 0.01$
$1 - \mathrm{P}(W \le 232) < 0.01$
$\mathrm{P}(W \le 232) > 0.99$
Using GC,
When $n = 239$, $\mathrm{P}(W \le 232) = 0.998 > 0.99$
When $n = 240$, $\mathrm{P}(W \le 232) = 0.995 > 0.99$
When $n = 241$, $\mathrm{P}(W \le 232) = 0.989 < 0.99$
When $n = 242$, $\mathrm{P}(W \le 232) = 0.977 < 0.99$
Hence the maximum reservations that should be accepted is 240.

(iii)
$Y \sim$ number of flights which is overbooked, out of 7.
$Y \sim \mathrm{B}(7\ ,\ 0.118761)$
$\mathrm{P}(Y = 0) = 0.41272 \approx 0.413$ (3 s.f.)

(iv)
Since n = 52 is large, by CLT,

$$\overline{Y} \sim N\left(\mathrm{E}(Y), \frac{\mathrm{Var}(Y)}{52}\right) \text{ approximately}$$

$$\overline{Y} \sim N\left((7)(0.118761), \frac{(7)(0.118761)(0.881239)}{52}\right) \text{ approximately}$$

$$\overline{Y} \sim N\left(0.831327, \frac{0.732598}{52}\right) \text{ approximately}$$

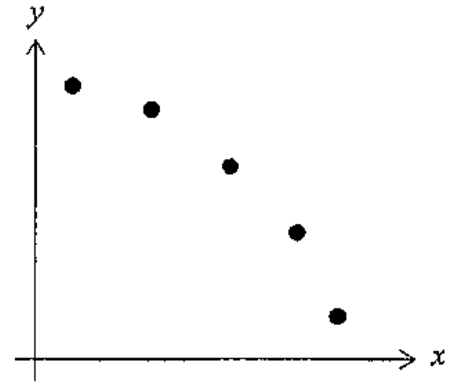
$$\overline{Y} \sim N\left(0.831327, 0.014088\right) \text{ approximately}$$

$\mathrm{P}(\overline{Y} \le 1) = 0.9223521 \approx 0.922$ (3 s.f.)

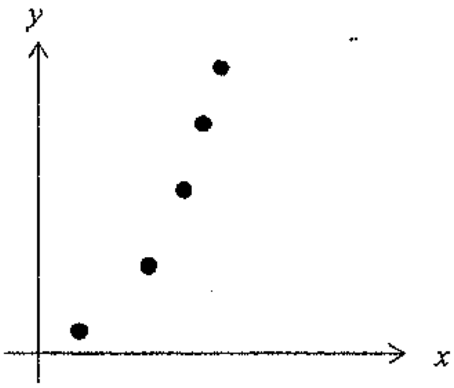
Q12

(i)

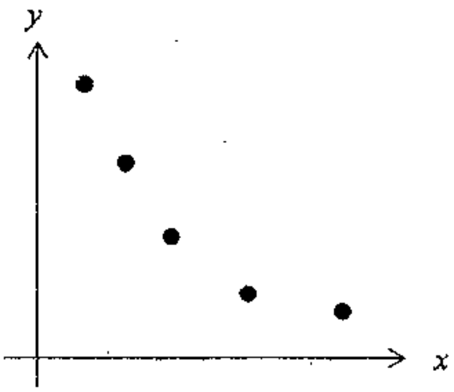
$y = p + qx^2$



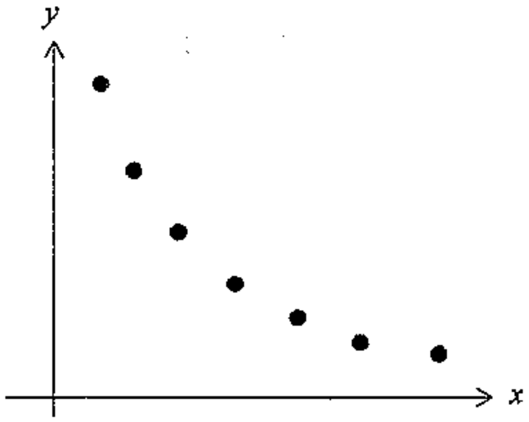
$y = r + se^x$



$y = t + \frac{u}{x}$



(ii)



(iii)

As x increases, y decreases at a decreasing rate. Hence, model (C) is the most appropriate.
Using GC, $r = 0.984$

(iv)

Equation of regression line: $y = 69.425 + \dfrac{5.75555}{x} \approx 69.4 + \dfrac{5.76}{x}$

When $x = 10$,

$y = 69.425 + \dfrac{5.75555}{10} = 70.0$

(iv)

Replace x with $\dfrac{x}{7}$,

New equation:

$y = 69.425 + \dfrac{5.75555}{\frac{x}{7}} \approx 69.4 + \dfrac{40.3}{x}$