9

RAFFLES INSTITUTION
2021 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 21

MATHEMATICS **9758/01**

Paper 1 **3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

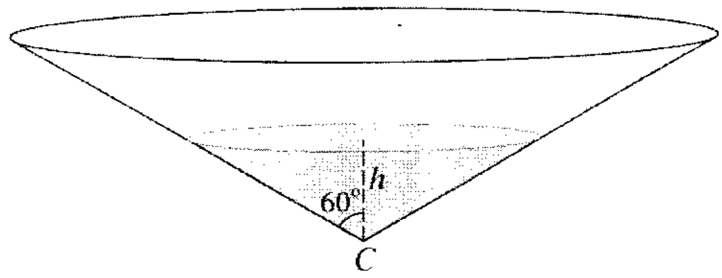
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE					
Q1	Q2	Q3	Q4	Q5	Q6
4	4	8	8	9	9
Q7	Q8	Q9	Q10	Q11	Total
11	11	12	12	12	100

RAFFLES INSTITUTION
Mathematics Department

1 Given the polynomial x^4+ax^2+bx+c has a factor $(x-2)$ and gives remainders 12 and 26 when divided by $(x-3)$ and $(x-4)$ respectively, find the values of a, b and c. [4]

2



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time $t=0$, the tank is filled with 94π cm^3 of water. At this instant, a tap at C is turned on and water begins to flow out at a constant rate of 2π cm^{3}s^{-1}. Denoting h cm as the depth of water at time t s, find the rate of decrease of h when $t=15$, leaving your answer in exact form. [4]

[The volume V of a cone of vertical height h and base radius r is given by $V=\frac{1}{3}\pi r^2 h$.]

3 **(a)** Find $\int x\tan^{-1}x\,\mathrm{d}x$. [3]

(b) **(i)** Using the substitution $u=\frac{1}{x}$, or otherwise, find $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$. [2]

(ii) Given that n is a positive integer, evaluate the integral $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$,

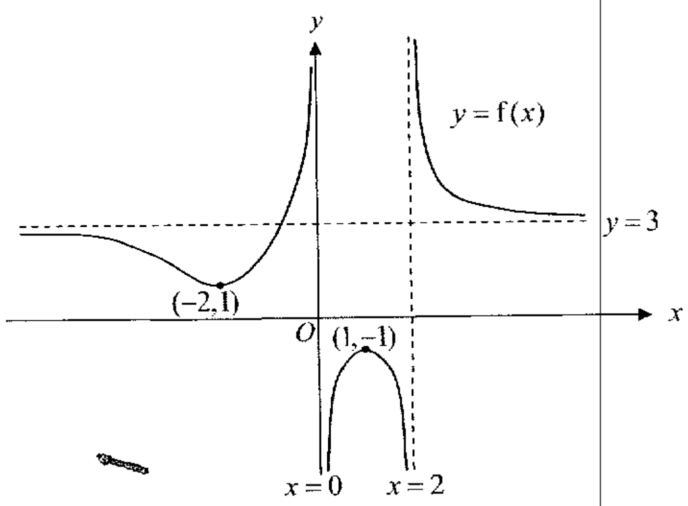
giving your answer in the form $a\pi$, where the possible values of a are to be determined. [3]

4 **(a)** Show that $(2r+1)^3-(2r-1)^3=12kr^2+k$, where k is a constant to be determined. Use this result to find $\sum_{r=1}^{n} r^2$, giving your answer in the form $pn(qn+1)(2qn+1)$ where p and q are constants to be determined. [5]

(b) Raabe's test states that a series of positive terms of the form $\sum_{r=1}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]>1$, and diverges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]<1$.

When $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]=1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=1}^{\infty}\frac{1}{r^3}$ converges. [3]

5 **(a)** The graph of $y=f(x)$ is shown below.



On separate diagrams, sketch the following graphs, indicating clearly the key features.

(i) $y=f(1-x)$, [3]

(ii) $y=f'(x)$. [3]

(b) State a sequence of transformations which transform the graph of $y=\ln\left(1-\frac{x}{2}\right)$ onto the graph of $y=\ln\left(\frac{2}{1-x}\right)$. [3]

6 **Do not use a calculator in answering this question.**

(a) Show that $z = 2\text{i}$ is a root of the equation $z^3 + 2z + 4\text{i} = 0$. [2]
Hence find the other roots. [3]

(b) Let $w_1 = -\dfrac{\sqrt{6}}{2} + \dfrac{\sqrt{2}}{2}\text{i}$ and $w_2 = 1 + \text{i}$.

Find the smallest positive integer n such that $\arg\left(\dfrac{w_2}{w_1}\right)^n = -\dfrac{\pi}{2}$. [4]

7 A curve C has parametric equations

$$x = \sin t\,,\quad y = \frac{1}{3}\cos t\,,\quad \text{for } -\pi \le t \le \frac{\pi}{4}.$$

(i) Find the equation of the normal to C at the point P with parameter $t = p$. [3]

(ii) The normal to C at the point when $t = -\dfrac{\pi}{4}$ cuts the curve again at point A. Find the coordinates of point A, correct to 2 decimal places. [4]

(iii) Sketch the graph of C, giving the coordinates of the end points in exact form. [2]

(iv) Find the area of the region bounded by C, the x-axis and the lines $x = 0$ and $x = \dfrac{1}{\sqrt{2}}$. [2]

8 **(i)** The curve G has equation $y = \dfrac{1}{1+x^2}$. Sketch the graph of G, stating the equation(s) of any asymptote(s) and the coordinates of any turning point(s). [2]

(ii) The line l intersects G at $x = 0$ and is tangential to G at the point (c, d), where $c > 0$. Find c and d, and determine the equation of l. [4]

Let R denote the region bounded by G, the x-axis and the lines $x = 0$ and $x = 1$.

(iii) By comparing the area of R and the area of the trapezoidal region between l and the x-axis for $0 \le x \le 1$, show that $\pi > 3$. [2]

(iv) By considering the volume of revolution of a suitable region rotated through 2π radians about the y-axis, show that $\ln 2 > \dfrac{2}{3}$. [3]

9 The equations of a plane p_1 and a line l are shown below:

$$p_1: \quad \mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10,$$

$$l: \quad \frac{x+1}{3} = z+4, \ y = 1.$$

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

(i) Find the coordinates of the foot of perpendicular, N, from A to p_1. [4]

(ii) Find the position vector of the point B which is the reflection of A in p_1. [2]

(iii) Hence, or otherwise, find an equation of the line l', the reflection of l in p_1. [4]

(iv) Another plane, p_2, contains B and is parallel to p_1. Determine the exact distance between p_1 and p_2. [2]

10 Bob purchases a house and takes a loan of \$$A$ from a bank. The sum of money owed to the bank t months after taking the loan is denoted by \$$x$. Both x and t are taken to be continuous variables. The sum of money owed to the bank increases, due to interest, at a rate proportional to the sum owed and decreases at a constant rate r as Bob repays the bank.
When $x = a$, interest and repayment balance. Write down a differential equation relating x and t, and solve it to give x in terms of t, r, a and A. [8]

State the condition under which the sum of money owed to the bank is repaid in a finite time T months, justifying your answer. Show that $T = \frac{a}{r}\ln\left(\frac{a}{a-A}\right)$. [4]

11 [It is given that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.]

Craft drinks have been gaining popularity in the beverage industry in recent years. These drinks are usually freshly made and served cold, with much attention given to the ingredients that make up the drinks and the entire process of preparation.

Ice is a very important ingredient in the making of a craft drink as it affects two crucial components: the temperature and the dilution of the drink. Hence, great emphasis is placed on the shapes of the ice, as different shapes will offer different surface areas and thus have a direct impact on the taste of the drink.

An ice manufacturer, who specialises in producing cylindrical shaped ice suitable for craft drinks served in tall glasses, wants to find out information about the surface area of the cylindrical shaped ice he produces.

(i) A piece of cylindrical shaped ice has radius r, height h and a fixed volume V. Show that its surface area, S, is given by $2\pi r^2 + \frac{2V}{r}$. [2]

(ii) Use differentiation to find, in terms of V, the minimum value of S, proving that it is a minimum. You are to give your answer in the form $k\left(m\pi V^m\right)^{\frac{1}{k}}$, where k and m are positive integers to be found. Find also the ratio $r : h$ that gives this minimum value of S. [7]

There has been a growing trend to use one large piece of ice for craft drinks to create a better drinking experience for the customers. Spherical shaped ice is considered ideal as it can keep the drink at a constant cold temperature with minimal dilution.

(iii) For the minimum value of S found in part **(ii)**, show that the volume of the largest spherical shaped ice that can be carved out is $\frac{m}{k}V$, where k and m are the same integers found in part **(ii)**. [2]

(iv) State, giving a reason, whether the manufacturer should proceed to carve out spherical shaped ice from the existing cylindrical shaped ice produced. [1]

RAFFLES INSTITUTION
2021 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 21

MATHEMATICS **9758/02**

Paper 2 **3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

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FOR EXAMINER'S USE

SECTION A: PURE MATHEMATICS

Q1	Q2	Q3	Q4	Total
/8	/10	/10	/12	/40

SECTION B: PROBABILITY AND STATISTICS

Q5	Q6	Q7	Q8	Q9	Q10	Total
/6	/8	/9	/12	/12	/13	/60

TOTAL
/100

RAFFLES INSTITUTION
Mathematics Department

Section A: Pure Mathematics [40 marks]

1 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{(x-1)^2}, \quad x \in \mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{1}{2-x}, \quad x \in \mathbb{R}, \quad 1 \le x < 2.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. [1]

(ii) If the domain of f is restricted to $x \ge k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x \ge k$, using the value of k found in part **(ii)**.

(iii) Find $\mathrm{g}^{-1}(x)$ and show that the composite function $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists. [4]

(iv) Find the range of $\mathrm{g}^{-1}\mathrm{f}^{-1}$. [1]

2 **(a)** Three consecutive terms of a decreasing geometric progression has a product of 5832. If the first number is reduced by 24, these 3 numbers in the same order will form an arithmetic progression. Find the three terms of the geometric progression. [5]

(b) The fractal called Sierpiński Triangle is depicted below. Fig. 1 shows an equilateral triangle of side 1. In stage 1, the triangle in Fig. 1 is divided into four smaller *identical* equilateral triangles and the middle triangle is removed to give the triangle shown in Fig. 2. In stage 2, the remaining three equilateral triangles in Fig. 2 are each divided into four smaller *identical* equilateral triangles and the middle triangles are removed to give the triangle shown in Fig. 3 and the process continues.

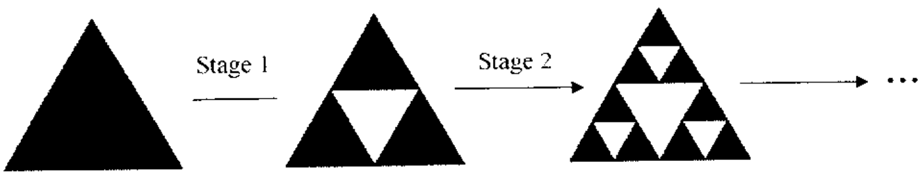


Fig. 1 **Fig. 2** **Fig. 3**

Let T_n be the total area of triangles removed after n stages of the process.

(i) Show that $T_1 = \frac{\sqrt{3}}{16}$. [1]

(ii) Find T_{10}. [3]

(iii) State the exact value of $\lim_{n \to \infty} T_n$. [1]

3 It is given that $\ln y=\sqrt{1+8e^x}$.

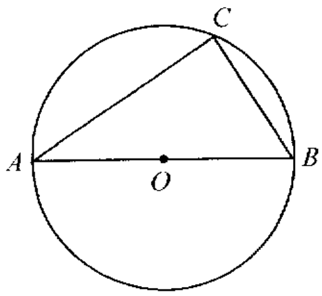
(i) Show that $(\ln y)\frac{dy}{dx}=4ye^x$. [1]

(ii) Show that the value of $\frac{d^2y}{dx^2}$ when $x=0$ is $\frac{68}{27}e^3$. [4]

(iii) Hence find the Maclaurin series for $e^{\sqrt{1+8e^x}}$ up to and including the term in x^2. [2]

(iv) Denoting the answer found in part **(iii)** as $g(x)$, find the set of values of x for which $g(x)$ is within ± 0.5 of the value of $e^{\sqrt{1+8e^x}}$. [3]

4 **(a)** **(i)**



Referred to the origin O, points A, B and C have position vectors **a**, **b** and **c** respectively. The three points lie on a circle with centre O and diameter AB (see diagram).
Using a suitable scalar product, show that the angle ACB is $90°$. [4]

(ii) The variable vector **r** satisfies the equation $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$. Describe the set of vectors **r** geometrically. [2]

(b) **(i)** The variable vector **r** satisfies the equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}$, where **m** and **n** are constant vectors. Describe the set of vectors **r** geometrically.
Give the geometrical meaning of $|\mathbf{m}\cdot\mathbf{n}|$ if **n** is a unit vector. [2]

(ii) The plane π passes through the points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ where x, y and z are non-zero constants. It is given that d is the perpendicular distance from the origin to π. Show, by finding the normal of π, or otherwise, that $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2}$. [4]

Section B: Probability and Statistics [60 marks]

5 For events A and B it is given that $\mathrm{P}(A) = 0.3$, $\mathrm{P}(B|A) = 0.4$ and $\mathrm{P}(A' \cap B') = 0.15$. Find

(i) $\mathrm{P}(A \cup B)$, [1]

(ii) $\mathrm{P}(B)$, [3]

(iii) $\mathrm{P}(A|B')$. [2]

6 The recruitment manager of the private car hire company, I-ber, claims that the mean weekly earnings of a full-time driver is \$980. The managing director suspects that the mean weekly earnings is less than \$980 and he instructs the recruitment manager to carry out a hypothesis test on a sample of drivers. It is given that the population standard deviation of the weekly earnings is \$88.

(i) State suitable hypotheses for the test, defining any symbols that you use. [2]

The recruitment manager takes a random sample of 10 drivers. He finds that the weekly earnings in dollars, are as follows.

942 950 905 1003 883 987 924 920 913 968

(ii) Find the mean weekly earnings of the sample of these 10 drivers. Carry out the test, at 5% level of significance, for the recruitment manager. Give your conclusion in context and state a necessary assumption for the test to be valid. [5]

(iii) Find the smallest level of significance at which the test would result in rejection of the null hypothesis, giving your answer correct to 1 decimal place. [1]

7 **In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A company sells hand sanitiser in bottles of two sizes – small and large. The amounts, in ml, of hand sanitiser in the small and large bottles, are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard deviation
Small bottles	108	5
Large bottles	510	σ

(i) Find the probability that the amount of hand sanitiser in a randomly chosen small bottle is less than 100 ml. [1]

(ii) During a quality control check on a batch of small bottles of hand sanitiser, 100 small bottles are randomly chosen to be inspected by an officer one at a time. Once he finds five bottles, each with amount of hand sanitiser less than 100 ml, that batch will be rejected. Find the probability that he had to check through all 100 bottles to reject that batch. [2]

(iii) Given that the amount of hand sanitiser in 85% of the large bottles lie within 9 ml of the mean, find σ. [3]

(iv) Given instead that $\sigma = 6$, find the probability that the amount of hand sanitiser in a randomly chosen large bottle is less than five times the amount of hand sanitiser in a randomly chosen small bottle. [3]

8 This question is about arrangements of all eight letters in the word IMMUNITY.

(i) Show that the number of different arrangements of the eight letters that can be made is 10080. [1]

(ii) Find the number of different arrangements that can be made with no two vowels next to each other. [3]

One of the 10080 arrangements in part **(i)** is randomly chosen.
Let A denote the event that the two I's are next to each other and let B denote the event that the two M's are next to each other.

(iii) Determine, with a reason, whether A and B are

(a) mutually exclusive, [1]

(b) independent. [3]

(iv) Find the probability that the chosen arrangement contains no two adjacent letters that are the same. [4]

9 A bag initially contains 3 red balls and 3 black balls. Whenever a red ball is drawn from the bag, it is put back into the bag together with an extra red ball. Whenever a black ball is drawn from the bag, it is not put back into the bag and no extra balls are added.

Isaac draws n balls from the bag, one after another, where $n \in \mathbb{Z}^+$, and R denotes the number of red balls out of the n balls drawn.

(a) Give two reasons why R cannot be modelled using a Binomial distribution. [2]

(b) For $n = 3$, find

(i) $\mathrm{P}(R \geq 1)$, [2]

(ii) the probability that the first ball drawn is black given that at least 1 of the 3 balls drawn is red. [3]

(c) For $n = 31$, show that $\mathrm{P}(R = 31) = \dfrac{1}{714}$. [2]

(d) Isaac wins 100 dollars for each red ball he draws if all the balls he draws from the bag are red, and does not win any money otherwise. What is the maximum amount of money Isaac would win if the probability of all the balls he draws are red exceeds 0.0001? [3]

10 A bag contains four balls numbered 1, 2, 3 and 4. In a game, a ball is drawn at random from the bag and then a fair coin is tossed a number of times that is equal to the number shown on the ball drawn. The random variable X is the number of heads recorded.

(i) Show that $\mathrm{P}(X = 0) = \dfrac{15}{64}$. Find $\mathrm{P}(X = x)$ for all other possible values of x. [5]

(ii) Denoting the expectation and variance of X by μ and σ^2 respectively, find $\mathrm{P}(X > \mu)$ and show that $\sigma^2 = \dfrac{15}{16}$. [3]

Adam plays this game 10 times.

(iii) Find the probability that there are at least two games with at least 2 heads recorded. [2]

Bill plays this game 50 times.

(iv) Using a suitable approximation, estimate the probability that the average number of heads recorded is less than 1. [3]

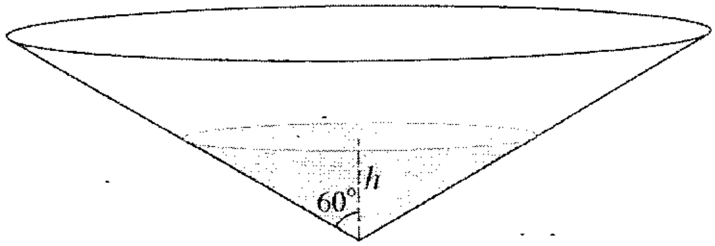
RAFFLES INSTITUTION
H2 Mathematics (9758)
2021 Year 6

2021 Year 6 H2 Math Preliminary Paper 1: Solutions with Comments

1 Given the polynomial x^4+ax^2+bx+c has a factor $(x-2)$ and gives remainders 12 and 26 when divided by $(x-3)$ and $(x-4)$ respectively, find the values of a, b and c. [4]

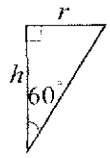
Solution		Comments
[4]	$\mathrm{f}(x)=x^4+ax^2+bx+c$ $\mathrm{f}(2)=0 \Rightarrow 16+4a+2b+c=0 \Rightarrow 4a+2b+c=-16$ ...(1) $\mathrm{f}(3)=12 \Rightarrow 81+9a+3b+c=12 \Rightarrow 9a+3b+c=-69$ ...(2) $\mathrm{f}(4)=26 \Rightarrow 256+16a+4b+c=26 \Rightarrow 16a+4b+c=-230$ ...(3) Solving (1), (2) and (3), $a=-54,\ b=217,\ c=-234$	The majority of the students did well for this question. However, there were some students who inefficiently did long division to arrive at the 3 equations, with some making errors along the way. This should not be an unfamiliar question as it makes use of factor and remainder theorems which students are assumed to have prior knowledge of.

2



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is $60°$, as shown in the diagram. At time $t = 0$, the tank is filled with 94π cm^3 of water. At this instant, a tap at C is turned on and water begins to flow out at a constant rate of 2π cm^{3}s^{-1}. Denoting h cm as the depth of water at time t s, find the rate of decrease of h when $t = 15$, leaving your answer in exact form. [4]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

Solution		**Comments**
[4]	Volume of water in the conical container at time t seconds, $V = \frac{1}{3}\pi r^2 h$ $\tan 60 = \frac{r}{h} \Rightarrow r = h\sqrt{3}$  Therefore $V = \pi h^3$ $\Rightarrow \frac{dV}{dt} = 3\pi h^2 \frac{dh}{dt}$ When $t = 15$, $V = 94\pi - (2\pi)(15) = 64\pi$, $\pi h^3 = 64\pi \Rightarrow h = 4$ $\therefore \frac{dh}{dt} = -2\pi \div 3\pi(4)^2 = -\frac{1}{24}$ $\therefore$ Rate at which h is decreasing at the instant when $t = 15$ is $\frac{1}{24}$ cms^{-1}.	Most students were able to handle the question with ease as it was something familiar that they had done before with some standard steps outlined in their lecture or revision notes. However, there were students who made the following **common errors**: 1. $h^3 = 64 \Rightarrow h = 8$ 2. $\frac{dh}{dt} = \frac{-2\pi}{3\pi h^2} = \frac{-2}{3}h^2$ 3. $\frac{dV}{dt} = 2\pi$ 4. Differentiating V without realising that both h and r are variables:

		Eg: $V=\frac{1}{3}\pi r^2 h$ $\Rightarrow \frac{dV}{dh}=\frac{1}{3}\pi r^2$ or $\frac{dV}{dr}=\frac{2}{3}\pi rh$ 5. Wrote either " $\frac{dh}{dt}=\frac{1}{24}$ " or "Rate of decrease of $h=-\frac{1}{24}$"

3 **(a)** Find $\int x \tan^{-1} x \, dx$. [3]

(b) **(i)** Using the substitution $u=\frac{1}{x}$, or otherwise, find $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx$. [2]

(ii) Given that n is a positive integer, evaluate the integral $\pi \int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx$,

giving your answer in the form $a\pi$, where the possible values of a are to be determined. [3]

Solutions		Comments
(a) **[3]**	$\int x\tan^{-1}x\,dx=\left(\frac{x^2}{2}\tan^{-1}x\right)-\int \frac{x^2}{2}\left(\frac{1}{1+x^2}\right)dx$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}\,dx$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[x-\tan^{-1}x\right]+c$ $=\frac{x^2+1}{2}\tan^{-1}x-\frac{x}{2}+c$	Most students were able to obtain the 1st line, but a number had difficulty proceeding on.

(b) **(i)** **[2]**	Given the substitution $u=\frac{1}{x}$, we have $\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{1}{x^2}$ $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=-\int \sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\mathrm{d}x$ $=-\int \sin u\,\mathrm{d}u$ $=\cos u+c$ $=\cos\left(\frac{1}{x}\right)+c$ OR $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=-\int\left(-\frac{1}{x^2}\right)\sin\left(\frac{1}{x}\right)\mathrm{d}x$ $=\cos\left(\frac{1}{x}\right)+c$	Generally this part was ok.
(b) **(ii)** **[3]**	$\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=\pi\left[\cos\left(\frac{1}{x}\right)\right]_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}}$ $=\pi\left[\cos(n\pi)-\cos((n+1)\pi)\right]$ If n is even, $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=\pi\left[1-(-1)\right]=2\pi \quad a=2$ If n is odd, $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=\pi\left[-1-1\right]=-2\pi \quad a=-2$ *OR* $a=2(-1)^n$	Quite a number of students wrote down $a=\pm 2$.

4 **(a)** Show that $(2r+1)^3-(2r-1)^3=12kr^2+k$, where k is a constant to be determined. Use this result to find $\sum_{r=1}^{n} r^2$, giving your answer in the form $pn(qn+1)(2qn+1)$ where p and q are constants to be determined. [5]

(b) Raabe's test states that a series of positive terms of the form $\sum_{r=1}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]>1$, and diverges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]<1$.

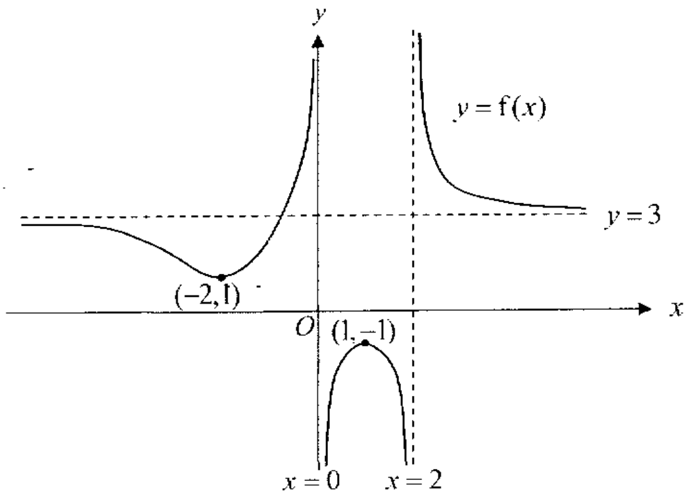
When $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]=1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=1}^{\infty}\frac{1}{r^3}$ converges. [3]

Solutions		**Comments**
(a) [5]	$(2r+1)^3-(2r-1)^3$ $=\left[8r^3+12r^2+6r+1\right]-\left[8r^3-12r^2+6r-1\right]$ $=24r^2+2$ $\therefore k=2$ OR $(2r+1)^3-(2r-1)^3$ $=\left[(2r+1)-(2r-1)\right]\left[(2r+1)^2+(2r+1)(2r-1)+(2r-1)^2\right]$ $=2\left[4r^2+4r+1+4r^2-1+4r^2-4r+1\right]$ $=2(12r^2+1)$ $=24r^2+2$ $\therefore k=2$	Very well done.

	$\sum_{r=1}^{n}\left(24r^2+2\right)=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$ $24\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}1=3^3 - 1^3$ $+5^3 - 3^3$ $+7^3 - 5^3$ $+ \ldots$ $+(2n-1)^3-(2n-3)^3$ $+(2n+1)^3-(2n-1)^3$ $=(2n+1)^3 - 1$ $24\sum_{r=1}^{n}r^2+2n=(2n+1)^3-1$ $\sum_{r=1}^{n}r^2=\frac{1}{24}\left[(2n+1)^3-(2n+1)\right]$ $=\frac{(2n+1)}{24}\left[(2n+1)^2-1\right]$ $=\frac{(2n+1)}{24}(2n+2)(2n)$ $=\frac{(2n+1)}{24}(2n+2)(2n)$ $=\frac{1}{6}n(n+1)(2n+1)$ $\therefore p=\frac{1}{6},\quad q=1$	Generally well done. Some common mistakes include having cubes of even numbers in the cancellation and wrongly concluding $\sum_{r=1}^{n}2=2$.

(b) [3]	Let $a_n = \frac{1}{n^3}$. $n\left(\frac{a_n}{a_{n+1}}-1\right) = n\left[\frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{(n+1)^3}\right)}-1\right]$ $= n\left[\frac{(n+1)^3}{n^3}-1\right]$ $= n\left(\frac{n^3+3n^2+3n+1-n^3}{n^3}\right)$ $= \frac{3n^2+3n+1}{n^2}$ $= 3+\frac{3}{n}+\frac{1}{n^2}$ $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left(3+\frac{3}{n}+\frac{1}{n^2}\right) = 3 > 1$. Therefore $\sum_{r=1}^{\infty}\frac{1}{r^3}$ converges.	Many students attempted to show that the expression $n\left(\frac{a_n}{a_{n+1}}-1\right) > 1$ and hence the limit is more than 1. However, this is not true in general. Some solutions had the n and r mixed up like $n\left(\frac{\frac{1}{r^3}}{\frac{1}{(r+1)^3}}-1\right)$. Better solutions come from those who worked out the value of the limit which is 3 by considering $n \to \infty$

5 **(a)** The graph of $y=\mathrm{f}(x)$ is shown below.

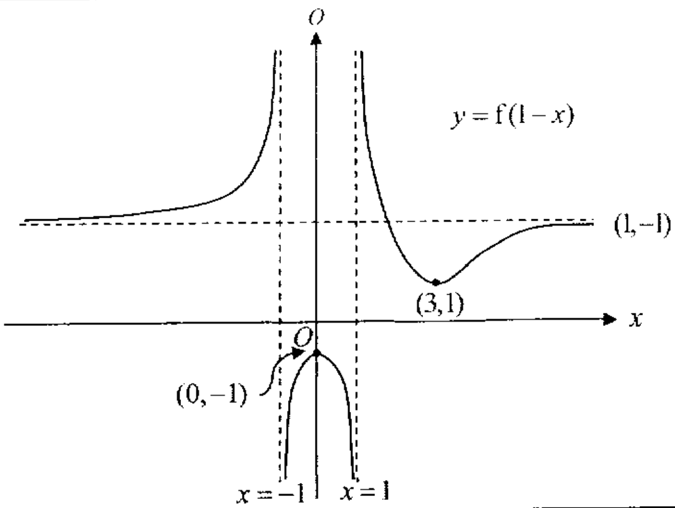


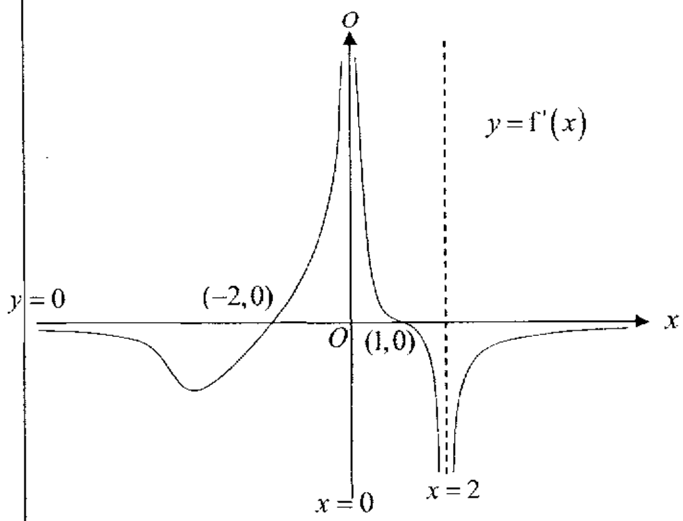
On separate diagrams, sketch the following graphs, indicating clearly the key features.

(i) $y=\mathrm{f}(1-x)$, [3]

(ii) $y=\mathrm{f}'(x)$. [3]

(b) State a sequence of transformations which transform the graph of $y=\ln\left(1-\frac{x}{2}\right)$ onto the graph of $y=\ln\left(\frac{2}{1-x}\right)$. [3]

Solutions		Comments
(a) **(i)** [3]		Most students correctly identify the two transformations involved, translation and reflection. However, a number of them did not transform the graph using the **correct sequence**. The correct method involves either "a translation of 1 unit in the **negative** *x*-direction followed by a reflection in ***y*-axis**"

		or "a reflection in **y-axis** followed by a translation of 1 unit in the **positive** *x*-direction". Some did not translate the graph in the correct direction or reflected in the wrong axis.
(a) (ii) [3]		Most students managed to derive the correct asymptotes and *x*-intercepts. However, a number of them drew part(s) of the graph in the wrong region(s) or had difficulties deriving the minimum turning point. Students need to label all the asymptotes including the axes.
(b) [3]	$y=\ln\left(1-\frac{x}{2}\right)\rightarrow y=\ln(1-x)\rightarrow y=-\ln(1-x)=\ln\left(\frac{1}{1-x}\right)$ $\rightarrow y=\ln\left(\frac{1}{1-x}\right)+\ln 2=\ln\left(\frac{2}{1-x}\right)$ Scale the graph of $y=\ln\left(1-\frac{x}{2}\right)$ parallel to the *x*-axis by factor $\frac{1}{2}$, followed by a reflection about the ***x*-axis**, followed by a translation of ln 2 units in the positive *y*-direction.	Most students managed to work out the correct "replacement" using algebraic manipulations but interpreted the corresponding transformation wrongly. A number of them used much longer method. Take note that students need to use

	OR $y=\ln\left(1-\frac{x}{2}\right)=\ln\left(\frac{2-x}{2}\right)\to y=-\ln\left(\frac{2-x}{2}\right)=\ln\left(\frac{2}{2-x}\right)$ $\to y=\ln\left(\frac{2}{1-x}\right)$ Reflect the graph of $y=\ln\left(1-\frac{x}{2}\right)$ about the ***x*-axis**, followed by a translation of 1 unit in the **negative** x-direction.	the correct terms/phrases to describe the transformations.

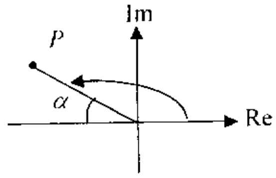
6 Do not use a calculator in answering this question.

(a) Show that $z=2\mathrm{i}$ is a root of the equation $z^3+2z+4\mathrm{i}=0$. [2]
Hence find the other roots. [3]

(b) Let $w_1=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$ and $w_2=1+\mathrm{i}$.

Find the smallest positive integer n such that $\arg\left(\frac{w_2}{w_1}\right)^n=-\frac{\pi}{2}$. [4]

Solutions		Comments
(a) [2]	Let $f(z)=z^3+2z+4\mathrm{i}$ $f(2\mathrm{i})=(2\mathrm{i})^3+2(2\mathrm{i})+4\mathrm{i}$ $=8\mathrm{i}^3+4\mathrm{i}+4\mathrm{i}$ $=-8\mathrm{i}+8\mathrm{i}$ $=0$ $z=2\mathrm{i}$ is a root of the equation $z^3+2z+4\mathrm{i}=0$.	As the question indicates a calculator is not to be used, all working and calculations need to be shown clearly. For (a), you should show the steps as shown in the solution.
[3]	$z^3+2z+4\mathrm{i}=0$ $(z-2\mathrm{i})(z^2+2\mathrm{i}z-2)=0$ $z=2\mathrm{i}$ or $z=\frac{-2\mathrm{i}\pm\sqrt{-4+8}}{2}$ $z=2\mathrm{i}$ or $z=-\mathrm{i}\pm1$ The other roots are $1-\mathrm{i},\ -1-\mathrm{i}$.	Do note that the equation $z^3+2z+4\mathrm{i}=0$ does not have real coefficients (4i). Hence to obtain the other 2 roots (cubic equation has 3 roots), you should do long division or compare coefficients using a quadratic factor. Splitting into 3 linear factors is tedious and unnecessary. After obtaining the quadratic equation with complex coefficients, you can complete squares or simply use the quadratic formula.
(b) [4]	We have $\arg(w_1)=\frac{5\pi}{6}$ and $\arg(w_2)=\frac{\pi}{4}$.	Please be careful as many swapped the two complex numbers w_1, w_2. Many students still cannot find the argument of a complex number not in the first quadrant (in this case w_1) correctly.

	Hence $\arg\left(\frac{w_2}{w_1}\right)=\arg(w_2)-\arg(w_1)$ $=\frac{\pi}{4}-\frac{5\pi}{6}$ $=-\frac{7\pi}{12}$ $\arg\left(\frac{w_2}{w_1}\right)^n=n\arg\left(\frac{w_2}{w_1}\right)$ $=-\frac{7n\pi}{12}$	For those who made this mistake, please follow the steps below: 1) Sketch quickly an argand diagram and locate the point representing the complex number w_1. It is in the 2nd quadrant.  $\arg(z)=\pi-\alpha$ 2) Find the basic angle $\tan\alpha=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}}=\frac{1}{\sqrt{3}}\Rightarrow\alpha=\frac{\pi}{6}$ 3) $\arg(w_1)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$.
	Hence we need to find the least positive integer n such that $\frac{-7n\pi}{12}=-\frac{\pi}{2}+m(2\pi)=\frac{(4m-1)\pi}{2},\ m\in\mathbb{Z}$. Rearranging, $n=\frac{6-24m}{7}=\frac{6(1-4m)}{7}$. **Method 1** Therefore we need to have an integer m such that $1-4m$ is positive (and thus negative m) and a multiple of 7. Checking through the negative integer values of m, we have $1-4m=5,9,13,17,\underline{21},\ldots$. The corresponding least value of n is therefore 18.	Note that since we are looking for the argument to be $-\frac{\pi}{2}$, the possible candidates are all angles that differ from it by a multiple of 2π. As the question said a calculator is not to be used, to justify that the smallest n is 18, you will need to explain why all the smaller n cannot work with clear working. (someone listed 18 such n, which while correct, is time consuming) Excellent responses often used some divisibility argument to reduce the cases to check, and subsequently calculating for each of the remaining cases the corresponding n, verifying clearly that they are **all** not integers.

	Method 2 m | $n=\dfrac{6(1-4m)}{7}$ -1 | $\frac{30}{7}$ -2 | $\frac{54}{7}$ -3 | $\frac{78}{7}$ -4 | $\frac{102}{7}$ -5 | 18 Hence smallest $n=18$, corresponding to when $m=-5$.	

7 A curve C has parametric equations

$$x=\sin t,\quad y=\frac{1}{3}\cos t,\ \text{for } -\pi \le t \le \frac{\pi}{4}.$$

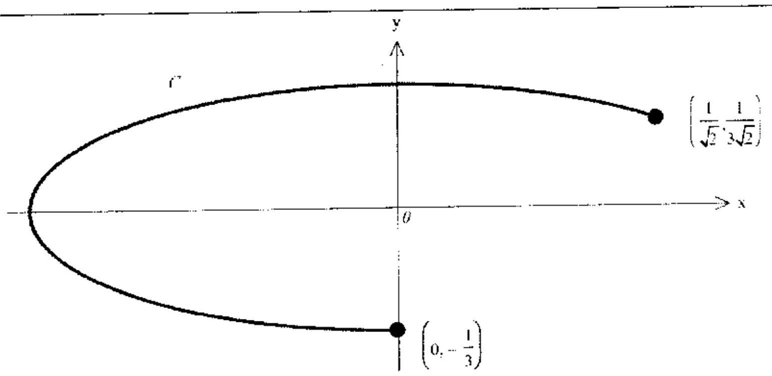
(i) Find the equation of the normal to C at the point P with parameter $t=p$. [3]

(ii) The normal to C at the point when $t=-\frac{\pi}{4}$ cuts the curve again at point A. Find the coordinates of point A, correct to 2 decimal places. [4]

(iii) Sketch the graph of C, giving the coordinates of the end points in exact form. [2]

(iv) Find the area of the region bounded by C, the x-axis and the lines $x=0$ and $x=\frac{1}{\sqrt{2}}$. [2]

Solutions		Comments
(i) [3]	$\frac{dx}{dt}=\cos t,\quad \frac{dy}{dt}=-\frac{1}{3}\sin t$ $\frac{dy}{dx}=-\frac{1}{3}\tan t$ At P, $\frac{dy}{dx}=-\frac{1}{3}\tan p$ Gradient of normal $=3\cot p$ Equation of normal at P: $y-\frac{1}{3}\cos p=(3\cot p)(x-\sin p)$ $y=(3\cot p)x-\frac{8}{3}\cos p$	Almost all candidates were able to differentiate the parametric equations to find $\frac{dy}{dx}$ but a significant number did not arrive at the correct expression for the gradient of the normal, either overlooking the negative sign in the relationship between the gradients of the tangent and normal or

		writing $-\dfrac{1}{\left(-\dfrac{1}{3}\tan t\right)}$ as $3\tan t$.
(ii) [4]	When $t=-\dfrac{\pi}{4}$, equation of normal: $y=(-3\cot\dfrac{\pi}{4})x-\dfrac{8}{3}\cos\dfrac{\pi}{4}$ $y=-3x-\dfrac{8}{3\sqrt{2}}$ ------ (*) For normal to cut C again, substitute $x=\sin t$, $y=\dfrac{1}{3}\cos t$ into (*) $\dfrac{1}{3}\cos t=-3\sin t-\dfrac{8}{3\sqrt{2}}$ $\dfrac{1}{3}\cos t+3\sin t=-\dfrac{8}{3\sqrt{2}}$ From GC, $t=-2.5775$ (4 d.p.) $x=\sin(-2.5775)=-0.53$ (2 d.p.) $y=\dfrac{1}{3}\cos(-2.5775)=-0.28$ (2 d.p.) The coordinates of point A is $(-0.53, -0.28)$.	Those candidates who answered this part well, usually approached it with the method as given in the solution, although there were some attempts which did not take into account the given range of values of t when solving.
(iii) [2]		Most candidates realised they had to adjust the window settings of the graphing calculator in order to obtain the part of the ellipse represented by the parametric equations corresponding to the given range of values of t.
(iv) [2]	Area of required region $=\int_0^{\frac{1}{\sqrt{2}}} y\,dx$ $=\int_0^{\frac{\pi}{4}} \dfrac{1}{3}\cos t\dfrac{dx}{dt}\,dt$ $=\dfrac{1}{3}\int_0^{\frac{\pi}{4}} \cos^2 t\,dt$ $=0.214$ units^2 (3sf)	Most candidates used the parametric equations to find the area, but not all correctly substituted $\dfrac{d(\sin t)}{dt}dt$ for dx. The most common incorrect answers involved candidates evaluating the integrals without converting the limits.

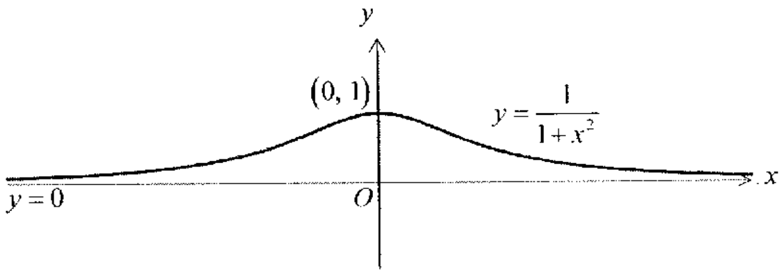
8 **(i)** The curve G has equation $y=\dfrac{1}{1+x^2}$. Sketch the graph of G, stating the equation(s) of any asymptote(s) and the coordinates of any turning point(s). [2]

(ii) The line l intersects G at $x=0$ and is tangential to G at the point (c, d), where $c>0$. Find c and d, and determine the equation of l. [4]

Let R denote the region bounded by G, the x-axis and the lines $x=0$ and $x=1$.

(iii) By comparing the area of R and the area of the trapezoidal region between l and the x-axis for $0 \le x \le 1$, show that $\pi > 3$. [2]

(iv) By considering the volume of revolution of a suitable region rotated through 2π radians about the y-axis, show that $\ln 2 > \dfrac{2}{3}$. [3]

Solutions		**Comments**
(i) [2]		Almost all the students are able to get this part correctly. Only a handful of students wrongly labelled the asymptote as $x=0$.
(ii) [4]	Now, $y=\dfrac{1}{1+x^2} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2x}{\left(1+x^2\right)^2}$. At (c, d), we have $d=\dfrac{1}{1+c^2}$ and $\dfrac{d-1}{c-0}=\dfrac{-2c}{\left(1+c^2\right)^2} \Rightarrow d=\dfrac{-2c^2}{\left(1+c^2\right)^2}+1$. Hence we have	Most students are able to do this part.

<table>
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$$\frac{-2c^2}{\left(1+c^2\right)^2}+1=\frac{1}{1+c^2}$$

$$\Rightarrow -2c^2+\left(1+c^2\right)^2=\left(1+c^2\right)$$

$$\Rightarrow c^4-c^2=0$$

$$\Rightarrow c^2\left(c^2-1\right)=0$$

$$\Rightarrow c=0 \text{ or } c=\pm 1.$$

Since $c>0$, $c=1$ and $d=\frac{1}{2}$.

Equation of l is $y-\frac{1}{2}=\frac{-2(1)}{\left(1+1^2\right)^2}(x-1)\Rightarrow y=-\frac{x}{2}+1$.

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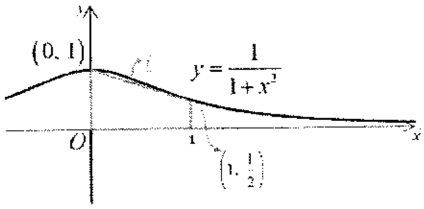
<tr>
<td>(iii)
[2]</td>
<td>

Area of $R=\int_0^1 \frac{1}{1+x^2}\,dx=\left[\tan^{-1} x\right]_0^1=\frac{\pi}{4}$.

Area of R > Area of trapezium

$$\Rightarrow \frac{\pi}{4}>\frac{1}{2}\left(1+\frac{1}{2}\right)(1)=\frac{3}{4}$$

$$\Rightarrow \pi>3 \text{ (Shown).}$$



Note:
Besides using the formula for area of trapezium, we can also use the following:

$$\int_0^1 -\frac{x}{2}+1\,dx=\frac{3}{4}.$$

</td>
<td>Students need to realize that they need to show $\pi>3$, hence the answer for area must be exact.</td>
</tr>
<tr>
<td>(iv)
[3]</td>
<td>

$$\pi\int_{\frac{1}{2}}^1 x^2\,dy=\pi\int_{\frac{1}{2}}^1 \frac{1}{y}-1\,dy$$

$$=\pi\left[\ln y-y\right]_{\frac{1}{2}}^1$$

$$=\pi\left[\left(\ln 1-1\right)-\left(\ln\frac{1}{2}-\frac{1}{2}\right)\right]$$

$$=\pi\left(-\ln\frac{1}{2}-\frac{1}{2}\right)=\pi\left(\ln 2-\frac{1}{2}\right).$$

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<td>Students who draw a diagram for (iii) will be able to see which volume to compare for (iv).</td>
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</table>

	Now, Volume obtained > Volume of cone with radius 1 and height $\frac{1}{2}$ $\Rightarrow \pi\int_{\frac{1}{2}}^{1} x^2 \, dy > \frac{1}{3}\pi\left(1^2\right)\left(\frac{1}{2}\right)$ $\Rightarrow \pi\left(\ln 2 - \frac{1}{2}\right) > \frac{\pi}{6}$ $\Rightarrow \ln 2 > \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$ (shown). Note: Besides using the formula for volume of cone, can also consider $\pi\int_{\frac{1}{2}}^{1} (2-2y)^2 \, dy = \frac{\pi}{6}$.	

9 The equations of a plane p_1 and a line l are shown below:

$$p_1: \quad \mathbf{r}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=10,$$

$$l: \quad \frac{x+1}{3}=z+4, \ y=1.$$

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

(i) Find the coordinates of the foot of perpendicular, N, from A to p_1. [4]

(ii) Find the position vector of the point B which is the reflection of A in p_1. [2]

(iii) Hence, or otherwise, find an equation of the line l', the reflection of l in p_1. [4]

(iv) Another plane, p_2, contains B and is parallel to p_1. Determine the exact distance between p_1 and p_2. [2]

<table>
<tr><th colspan="2">Solutions</th><th>Comments</th></tr>
<tr><td>(i)
[4]</td><td>Where $l_{AN}: \mathbf{r}=\begin{pmatrix}2\\1\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$ intersects $\pi:\mathbf{r}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=10$,
$\left[\begin{pmatrix}2\\1\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=10$
$(2-1-3)+\lambda(1+1+1)=10 \Rightarrow 3\lambda=12 \quad \Rightarrow \quad \lambda=4$
$\overrightarrow{ON}=\begin{pmatrix}2\\1\\-3\end{pmatrix}+4\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}6\\-3\\1\end{pmatrix}$
The coordinates of N is $(6,-3,1)$.</td><td>Most students did well for this question except some misinterpreted $\begin{pmatrix}2\\1\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\overrightarrow{AN}$ instead of $\overrightarrow{ON}$ and did not indicate final answer in coordinates form.</td></tr>
<tr><td>(ii)
[2]</td><td>$\overrightarrow{OB}=\overrightarrow{OA}+2\overrightarrow{AN}$
$=\overrightarrow{OA}+2\overrightarrow{ON}-2\overrightarrow{OA}$
$=2\overrightarrow{ON}-\overrightarrow{OA}$
$=2\begin{pmatrix}6\\-3\\1\end{pmatrix}-\begin{pmatrix}2\\1\\-3\end{pmatrix}=\begin{pmatrix}10\\-7\\5\end{pmatrix}$</td><td>Most students were able to either apply ratio theorem or use alternative vector form to find $\overrightarrow{OB}$, but were careless when computing the final answer.</td></tr>
<tr><td>(iii)
[4]</td><td>Equation of line $l: \mathbf{r}=\begin{pmatrix}-1\\1\\-4\end{pmatrix}+s\begin{pmatrix}3\\0\\1\end{pmatrix}, \quad s\in\mathbb{R}$</td><td>Most students were able to compute the equation of line l except those who made mistakes in identifying</td></tr>
</table>

	When $s=1$, $\mathbf{r}=\begin{pmatrix}-2\\1\\-3\end{pmatrix}=\overrightarrow{OA}$, so A lies on l Let point of intersection of l and p_1 be X. When l intersects p_1, $\left[\begin{pmatrix}-1\\1\\-4\end{pmatrix}+s\begin{pmatrix}3\\0\\1\end{pmatrix}\right].\begin{pmatrix}1\\-1\\1\end{pmatrix}=10 \Rightarrow s=4$ $\overrightarrow{OX}=\begin{pmatrix}11\\1\\0\end{pmatrix}$ Equation of reflected line $l':\mathbf{r}=\begin{pmatrix}10\\-7\\5\end{pmatrix}+\mu\left(\begin{pmatrix}10\\-7\\5\end{pmatrix}-\begin{pmatrix}11\\1\\0\end{pmatrix}\right)$ $l':\mathbf{r}=\begin{pmatrix}10\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}-1\\-8\\5\end{pmatrix}, \quad \mu\in\mathbb{R}$	$\begin{pmatrix}-1\\1\\-4\end{pmatrix}$ and $\begin{pmatrix}3\\0\\1\end{pmatrix}$. Some students failed to verify point A lies on line l which is essential in using point B to find direction vector in the given solution. Some students went through part (i) and (ii) approach to find the direction vector of l' which is lengthy and prone to careless mistake in computation.
(iv) **[2]**	Distance between p_1 and $p_2=\frac{1}{2}AB=\frac{1}{2}\left\|\begin{pmatrix}8\\-8\\8\end{pmatrix}\right\|=4\sqrt{3}$ OR Distance between p_1 and $p_2=BN=AN=\left\|4\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|=4\sqrt{3}$ OR $p_2:\mathbf{r}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}10\\-7\\5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=22$ Distance between p_1 and $p_2=\frac{22-10}{\sqrt{1^2+1^2+1^2}}=\frac{12}{\sqrt{3}}=4\sqrt{3}$ OR Distance between p_1 and p_2 $=\frac{\left\|\overrightarrow{XB}.\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|}{\sqrt{1^2+1^2+1^2}}=\frac{\left\|\begin{pmatrix}-1\\-8\\5\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|}{\sqrt{1^2+1^2+1^2}}=\frac{12}{\sqrt{3}}=4\sqrt{3}$	Most students were able to compute the distance either via BN and AN. There were some who applied length of projection onto normal vector which is acceptable as well.

10 Bob purchases a house and takes a loan of $\$A$ from a bank. The sum of money owed to the bank t months after taking the loan is denoted by $\$x$. Both x and t are taken to be continuous variables. The sum of money owed to the bank increases, due to interest, at a rate proportional to the sum owed and decreases at a constant rate r as Bob repays the bank.
When $x = a$, interest and repayment balance. Write down a differential equation relating x and t, and solve it to give x in terms of t, r, a and A. [8]

State the condition under which the sum of money owed to the bank is repaid in a finite time T months, justifying your answer. Show that $T = \frac{a}{r}\ln\left(\frac{a}{a-A}\right)$. [4]

Solutions		**Comments**
[8]	Since the sum of money owed to the bank increases at a rate proportional to the sum owed and Bob repays the bank at a constant rate r, $\frac{dx}{dt} = kx - r$, where $k > 0$. When $x = a$, interest and repayment balance. Then $\frac{dx}{dt} = 0 = ka - r \Rightarrow k = \frac{r}{a}$ Therefore $\frac{dx}{dt} = \frac{r}{a}(x) - r = \frac{r}{a}(x-a)$	Most students were able to form the differential equation correctly. It is recommended to find k using the information that $\frac{dx}{dt} = 0$ at $x = a$ and simplify the differential equation before solving it.
	$\frac{dx}{dt} = \frac{r}{a}(x-a)$ $\int \frac{1}{x-a}\,dx = \int \frac{r}{a}\,dt$ $\ln\lvert x-a\rvert = \frac{rt}{a} + C \quad , C \in \mathbb{R}$ $\lvert x-a\rvert = e^{\frac{rt}{a}}e^{C}$ $x - a = Be^{\frac{rt}{a}} \quad$ where $B = \pm e^{C}$ $x = Be^{\frac{rt}{a}} + a$	A handful of students forgot to include the absolute sign on $x-a$ in the natural log when performing integration on LHS, or failed to consider $\pm$ when taking exponential on both side in the next line.
	When $t = 0, x = A$, $A = B + a \Rightarrow B = A - a$ $x = (A-a)e^{\frac{rt}{a}} + a$	There were some students who inappropriately used A to denote the arbitrary constant, overlooking the fact that the letter is used to denote the amount of loan $\$A$ in the question. This often resulted in the solution which is in terms of A to be wrong.

[4]	For the loan to be repaid in a finite time T, $x=(A-a)e^{\frac{rt}{a}}+a$ must be a decreasing function as t increases. So $A-a<0 \Rightarrow A<a$ When the loan is repaid, $x=0$. $0=(A-a)e^{\frac{rT}{a}}+a$ $\Rightarrow \frac{a}{a-A}=e^{\frac{rT}{a}}$ $\Rightarrow T=\frac{a}{r}\ln\left(\frac{a}{a-A}\right)$ (Shown)	Even though most students were able to state $x=0$ when loan is repaid, many were not able to prove the statement because they did not obtain the correct solution from the previous part.

11 [It is given that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.]

Craft drinks have been gaining popularity in the beverage industry in recent years. These drinks are usually freshly made and served cold, with much attention given to the ingredients that make up the drinks and the entire process of preparation.

Ice is a very important ingredient in the making of a craft drink as it affects two crucial components: the temperature and the dilution of the drink. Hence, great emphasis is placed on the shapes of the ice, as different shapes will offer different surface areas and thus have a direct impact on the taste of the drink.

An ice manufacturer, who specialises in producing cylindrical shaped ice suitable for craft drinks served in tall glasses, wants to find out information about the surface area of the cylindrical shaped ice he produces.

(i) A piece of cylindrical shaped ice has radius r, height h and a fixed volume V. Show that its surface area, S, is given by $2\pi r^2 + \frac{2V}{r}$. [2]

(ii) Use differentiation to find, in terms of V, the minimum value of S, proving that it is a minimum. You are to give your answer in the form $k\left(m\pi V^m\right)^{\frac{1}{k}}$, where k and m are positive integers to be found. Find also the ratio $r : h$ that gives this minimum value of S. [7]

There has been a growing trend to use one large piece of ice for craft drinks to create a better drinking experience for the customers. Spherical shaped ice is considered ideal as it can keep the drink at a constant cold temperature with minimal dilution.

(iii) For the minimum value of S found in part **(ii)**, show that the volume of the largest spherical shaped ice that can be carved out is $\frac{m}{k}V$, where k and m are the same integers found in part **(ii)**. [2]

(iv) State, giving a reason, whether the manufacturer should proceed to carve out spherical shaped ice from the existing cylindrical shaped ice produced. [1]

Solutions		Comments
(i) [2]	$V=\pi r^2 h \Rightarrow h=\dfrac{V}{\pi r^2}$. Now, $S=2\pi r^2+2\pi rh$ $=2\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)$ $=2\pi r^2+\dfrac{2V}{r}$ (Shown).	Most students did well for part (i) and were able to prove the result. However some students did not show the substitution steps in detail.
(ii) [7]	$S=2\pi r^2+\dfrac{2V}{r}\Rightarrow\dfrac{dS}{dr}=4\pi r-\dfrac{2V}{r^2}$. $\dfrac{dS}{dr}=0\Rightarrow 4\pi r-\dfrac{2V}{r^2}=0$ $\Rightarrow 4\pi r^3-2V=0$ $\Rightarrow r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$. $\dfrac{d^2S}{dr^2}=4\pi+\dfrac{4V}{r^3}\Rightarrow\left.\dfrac{d^2S}{dr^2}\right\|_{r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}}=4\pi+\dfrac{4V}{\left(\frac{V}{2\pi}\right)}=12\pi>0$. So S is minimum when $r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$. $\therefore S=2\pi r^2+\dfrac{2V}{r}$ $=2\pi\left(\dfrac{V}{2\pi}\right)^{\frac{2}{3}}+\dfrac{2V}{\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}}$ $=(2\pi)^{\frac{1}{3}}V^{\frac{2}{3}}+2^{\frac{4}{3}}\pi^{\frac{1}{3}}V^{\frac{2}{3}}$ $=\left(2\pi V^2\right)^{\frac{1}{3}}(1+2)$ $=3\left(2\pi V^2\right)^{\frac{1}{3}}$, where $k=3$ and $m=2$. When $r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$, $\dfrac{r}{h}=\dfrac{r}{\left(\frac{V}{\pi r^2}\right)}=\dfrac{\pi r^3}{V}=\dfrac{\pi\left(\frac{V}{2\pi}\right)}{V}=\dfrac{1}{2}$.	Most students were able to perform differentiation well and find the minimum value of S and r. However, there were students who made the following common errors: 1. Letting $\dfrac{dV}{dr}=0$ or $\dfrac{dS}{dV}=0$ instead of $\dfrac{dS}{dr}=0$, when V is already stated clearly in the question as a **fixed** volume. 2. A handful of students did not prove that S is minimum by either 2nd or 1st derivative test. 3. Among students who used 1st derivative test to prove min, many did not factorize or break down $\dfrac{dS}{dr}$ before concluding that S is a min 4. Quite a handful of students did not state the final values of k and m Majority of students had difficulty finding the ratio of

	Therefore $r:h = 1:2$.	$r:h$. Among those who found it correctly, many translated to the wrong ratio. Eg. $h=2r$ should obtain a ratio of $r:h=1:2$, however it was written as $r:h=2:1$. Many students gave the final answer in fraction form too when the question requested for ratio form.
(iii) [2]	Largest spherical shaped ice that can be carved out has radius, $R=r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$, since when S is minimum, $h=2r$. Hence the volume of the largest spherical ice is $\frac{4}{3}\pi R^3=\frac{4}{3}\pi\left(\frac{V}{2\pi}\right)=\frac{2V}{3}$ (Shown).	Students who attempted part (ii) correctly were mostly able to identify that $R=r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$ before finding the volume of the largest spherical ice. However, majority of students failed to give a proper justification on why $R=r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$ and show the relation between the spherical shaped ice which will be carved out from the existing cylindrical shaped ice. Some students also had the **misconception** that the spherical shaped ice shares the same surface area as the cylindrical shaped ice, which led to students equating the two surface areas to find the volume.
(iv) [1]	No, the manufacturer should not proceed as the spherical shaped ice has volume at least $\frac{2}{3}V$ and so $\frac{1}{3}$ of the volume of the cylindrical shaped ice will go to waste which is quite a lot. OR	Most students were unable to give a reason on why the manufacturer should proceed to carve out spherical shaped ice from the existing cylindrical shaped ice produced. Students should note that the reason is to be given from the ice manufacturer's viewpoint,

	Yes, the manufacturer should proceed even though the spherical shaped ice has volume at least $\frac{2}{3}V$ as the $\frac{1}{3}V$ of crushed ice that is leftover during carving can be used for other drinks which require crushed ice.	NOT from the beverage industry. Some common misconceptions include students giving reasons that less ice will make the drink less diluted etc. However, the dilution of the drink is not of any concern to the ice manufacturer as they do not produce the drink. Another misconception which students have is that since the spherical shaped ice is smaller than the cylindrical shaped ice, the manufacturer will be able to save cost by producing less ice. However, it is not the case as the spherical shaped ice will STILL have to be carved out from the existing cylindrical shaped ice, which means that the manufacturer will have to produce the cylindrical shaped ice in full first before carving out the smaller spherical shaped ice.

RAFFLES INSTITUTION
H2 Mathematics (9758)
2021 Year 6

2021 Year 6 H2 Math Preliminary Paper 2: Solutions with Comments

Section A: Pure Mathematics [40 marks]

1 Functions f and g are defined by

$$f : x \mapsto e^{(x-1)^2}, \quad x \in \mathbb{R},$$

$$g : x \mapsto \frac{1}{2-x}, \quad x \in \mathbb{R}, \quad 1 \le x < 2.$$

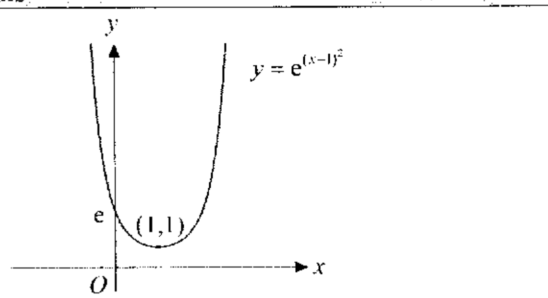
(i) Sketch the graph of $y = f(x)$. [1]

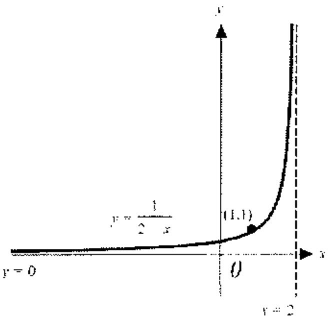
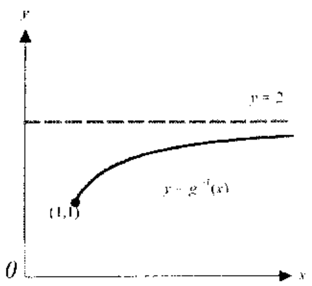
(ii) If the domain of f is restricted to $x \ge k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x \ge k$, using the value of k found in part **(ii)**.

(iii) Find $g^{-1}(x)$ and show that the composite function $g^{-1}f^{-1}$ exists. [4]

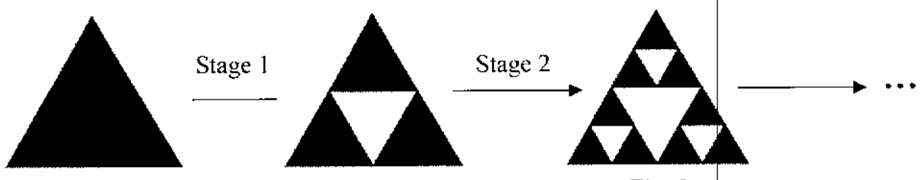
(iv) Find the range of $g^{-1}f^{-1}$. [1]

Solutions		**Comments**
(i) **[1]**		The graph and its properties can be easily obtained from the GC. However, a good number of students did not label either or both the y-intercept and minimum point. Students should also note that the graph is symmetrical about the line $x = 1$.
(ii) **[2]**	For f^{-1} to exist, f must be a one-one function. Least value of $k = 1$.	Many students elaborated the horizontal line test instead of stating that the condition is for f to be a 1-1 function. In this case, students have to take note of the precise phrasing "**every** horizontal line $y = k,\ k \in [1, \infty)$, cuts the graph of **f** at **one and only one point**."

(iii) [4]	Let $y=\dfrac{1}{2-x}$ $2-x=\dfrac{1}{y}$ $x=2-\dfrac{1}{y}$ $g^{-1}(x)=2-\dfrac{1}{x}$ For $g^{-1}f^{-1}$ to exist, $R_{f^{-1}}\subseteq D_{g^{-1}}$. $R_{f^{-1}}=D_f=[1,\infty)$ $D_{g^{-1}}=R_g=[1,\infty)$ Since $R_{f^{-1}}=D_{g^{-1}}$, $\therefore$ $g^{-1}f^{-1}$ exists.  	Most students managed to find $g^{-1}(x)$ correctly, with a very small number leaving their answers as $g^{-1}(x)=2-\dfrac{1}{y}$, which is incorrect. Most students could remember the condition to check for $g^{-1}f^{-1}$ to exist. Often mistakes were made in either finding $D_{g^{-1}}$ or keeping $D_f=\mathbb{R}$ (unrestricted).
(iv) [1]	$[1,\infty)\xrightarrow{f^{-1}}[1,\infty)\xrightarrow{g^{-1}}[1,2)$ $R_{g^{-1}f^{-1}}=[1,2)$ Note: $D_{f^{-1}}=R_f=[1,\infty)$, $R_{f^{-1}}=D_f=[1,\infty)$  **OR** Since $R_{f^{-1}}=D_{g^{-1}}$, $R_{g^{-1}f^{-1}}=R_{g^{-1}}=D_g=[1,2)$	Most students who used the arrow diagram successfully found $R_{g^{-1}f^{-1}}$. Students have to note the order of the functions involved and the horizontal asymptote in the graph of g^{-1}. A number of students also managed to recognize the relationship in alternative method, thus were able to state the answer correctly.

2 **(a)** Three consecutive terms of a decreasing geometric progression has a product of 5832. If the first number is reduced by 24, these 3 numbers in the same order will form an arithmetic progression. Find the three terms of the geometric progression. [5]

(b) The fractal called Sierpiński Triangle is depicted below. Fig. 1 shows an equilateral triangle of side 1. In stage 1, the triangle in Fig. 1 is divided into four smaller *identical* equilateral triangles and the middle triangle is removed to give the triangle shown in Fig. 2. In stage 2, the remaining three equilateral triangles in Fig. 2 are each divided into four smaller *identical* equilateral triangles and the middle triangles are removed to give the triangle shown in Fig. 3 and the process continues.



Let T_n be the total area of triangles removed after n stages of the process.

(i) Show that $T_1 = \frac{\sqrt{3}}{16}$. [1]

(ii) Find T_{10}. [3]

(iii) State the exact value of $\lim_{n\to\infty} T_n$. [1]

Solution		**Comments**
(a) [5]	Let the 3 numbers be $\frac{x}{r}$, x and xr, where x is the middle term and r is the common ratio. $\frac{x}{r}(x)(xr) = 5832$ $x^3 = 5832$ $x = 18$ If the first number is reduced by 24, it is now $\frac{x}{r} - 24$. Since the 3 numbers now form an AP, $\left(\frac{x}{r} - 24\right) - x = x - xr$ Substitute $x = 18$ into the above equation,	Most students were able to solve this part successfully, but the efficiency depends on what they wrote as the first 3 terms. Quite many let the 3 numbers be xr^n, xr^{n+1}, xr^{n+2}, and could not solve due to the extra variable n. They were able to get $xr^{n+1} = 18$ but

	$\left(\frac{18}{r}-24\right)-18=18-18r$ $3r^2-10r+3=0$ $(3r-1)(r-3)=0$ $\therefore r=\frac{1}{3}$ or $r=3$ (rejected $\because$ it is a decreasing GP, i.e. $0<r<1$) Thus, the original 3 numbers are 54, 18, 6.	could not solve it due to failure to recognise this as the second term.
(b) (i) [1]	Let A be the original area of the triangle in Fig. 1. $A=\frac{1}{2}(1)(1)\sin 60°$ $=\frac{\sqrt{3}}{4}$ $T_1=\frac{1}{4}A=\frac{\sqrt{3}}{16}$ (shown)	Students who could not solve this part generally were not able to sieve out the information that there are 4 smaller identical triangles in Figure 2. Many students resorted to finding the height of the triangle in Figure 1 to find the area, when a simple application of the formula suffice.
(ii) [3]	Area of triangle removed in stage $1=T_1$ Area of triangles removed in stage $2=\frac{3}{4}T_1$ Area of triangles removed in stage $3=\left(\frac{3}{4}\right)^2 T_1$ $\vdots$ Area of triangles removed in stage $n=\left(\frac{3}{4}\right)^{n-1} T_1$	Students who were able to identify a GP generally managed to get the answer. Many students sort of recognise that there is some addition of the area of triangles, and tried to use the sum of a GP

	Total area of triangles removed after 10 stages, T_{10} $=T_1+\frac{3}{4}T_1+\left(\frac{3}{4}\right)^2T_1+\cdots+\left(\frac{3}{4}\right)^9T_1$ $=\frac{T_1\left(1-\left(\frac{3}{4}\right)^{10}\right)}{1-\frac{3}{4}}$ $=0.409$ (3 s.f.)	formula. Not many left the part blank.
(iii) **[1]**	$\lim_{n\to\infty}T_n=\frac{T_1}{1-\frac{3}{4}}=\frac{\sqrt{3}}{4}$. **OR** $\lim_{n\to\infty}T_n=A=\frac{\sqrt{3}}{4}$	It is good to see that many students, while not able to solve the previous part, make an attempt at part (iii).

3 It is given that $\ln y=\sqrt{1+8e^x}$.

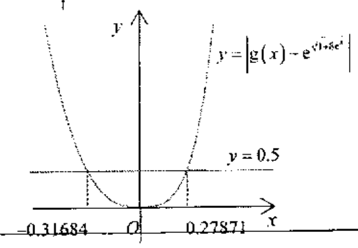
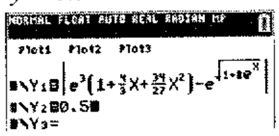
(i) Show that $(\ln y)\frac{dy}{dx}=4ye^x$. [1]

(ii) Show that the value of $\frac{d^2y}{dx^2}$ when $x=0$ is $\frac{68}{27}e^3$. [4]

(iii) Hence find the Maclaurin series for $e^{\sqrt{1+8e^x}}$ up to and including the term in x^2. [2]

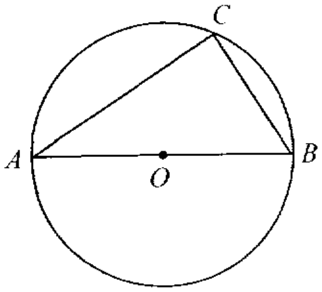
(iv) Denoting the answer found in part **(iii)** as $g(x)$, find the set of values of x for which $g(x)$ is within ±0.5 of the value of $e^{\sqrt{1+8e^x}}$. [3]

Solution		Comments
(i) **[1]**	$\ln y=\sqrt{1+8e^x}$ Differentiate w.r.t x, $\frac{1}{y}\frac{dy}{dx}=\frac{1}{2\sqrt{1+8e^x}}\left(8e^x\right)$ $\frac{1}{y}\frac{dy}{dx}=\frac{4e^x}{\ln y}$ $(\ln y)\frac{dy}{dx}=4ye^x$ (shown)	As this is a show question, clear working is vital. Students who did this question in another way should consider these 2 solutions provided as they are the most efficient.

	OR $\ln y=\sqrt{1+8e^{x}}$ $(\ln y)^2=1+8e^{x}$ Differentiate w.r.t x. $2(\ln y)\left(\frac{1}{y}\right)\frac{dy}{dx}=8e^{x}$ $(\ln y)\frac{dy}{dx}=4ye^{x}$ (shown)	
(ii) **[4]**	$(\ln y)\frac{dy}{dx}=4ye^{x}$ Differentiate w.r.t x. $(\ln y)\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2=4ye^{x}+4e^{x}\frac{dy}{dx}$ When $x=0$, $e^{x}=1$. $\ln y=3\Rightarrow y=e^3$ $3\frac{dy}{dx}=4e^3\Rightarrow\frac{dy}{dx}=\frac{4}{3}e^3$ $3\frac{d^2y}{dx^2}+\frac{1}{e^3}\left(\frac{4}{3}e^3\right)^2=4e^3+4\left(\frac{4}{3}e^3\right)$ $3\frac{d^2y}{dx^2}+\frac{16}{9}e^3=\frac{28}{3}e^3$ $\frac{d^2y}{dx^2}=\frac{1}{3}\left(\frac{28}{3}e^3-\frac{16}{9}e^3\right)=\frac{68}{27}e^3$ (shown)	One key to success in Maclaurin Series question is the method used to find the 2nd and 3rd derivatives. So always pay attention to the method used. The method shown here is the most efficient - **differentiate what you have been asked to show**. You are strongly encouraged to use this method and not any others which are inefficient.
(iii) **[2]**	$\ln y=\sqrt{1+8e^{x}}\Leftrightarrow y=e^{\sqrt{1+8e^{x}}}$ By Maclaurin Theorem, $e^{\sqrt{1-8e^{x}}}=e^3+\frac{4}{3}e^3x+\frac{68}{27}e^3\frac{x^2}{2!}+\dots$ $=e^3\left(1+\frac{4}{3}x+\frac{34}{27}x^2\right)+\dots$	
(iv) **[3]**	Let $g(x)=e^3\left(1+\frac{4}{3}x+\frac{34}{27}x^2\right)$. $\left\|g(x)-e^{\sqrt{1+8e^{x}}}\right\|<0.5$  	To solve the inequality, (i) sketch the graphs of $y=\left\|g(x)-e^{\sqrt{1+8e^{x}}}\right\|$ and $y=0.5$ 

	$x \in (-0.317, 0.279)$ (3 s.f.)	(ii) find the points of intersection (iii) write down the range of values that satisfy the inequality by looking at the graphs

4 **(a)** **(i)**



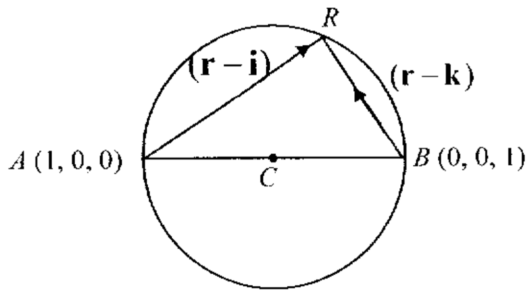
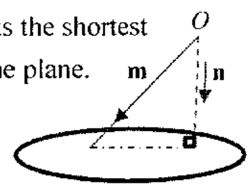
Referred to the origin O, points A, B and C have position vectors **a**, **b** and **c** respectively. The three points lie on a circle with centre O and diameter AB (see diagram).
Using a suitable scalar product, show that the angle ACB is $90°$. [4]

(ii) The variable vector **r** satisfies the equation $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$. Describe the set of vectors **r** geometrically. [2]

(b) **(i)** The variable vector **r** satisfies the equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}$, where **m** and **n** are constant vectors. Describe the set of vectors **r** geometrically.
Give the geometrical meaning of $|\mathbf{m}\cdot\mathbf{n}|$ if **n** is a unit vector. [2]

(ii) The plane π passes through the points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ where x, y and z are non-zero constants. It is given that d is the perpendicular distance from the origin to π. Show, by finding the normal of π, or otherwise, that $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2}$. [4]

Solution		**Comments**
(a) (i) [4]	$\overrightarrow{AC}\cdot\overrightarrow{BC}=(\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}+\mathbf{a})$ (since $\mathbf{b}=-\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{a}$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ $=\lvert\mathbf{c}\rvert^2-\lvert\mathbf{a}\rvert^2$ (since $\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{c}\rvert^2, \mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) $=0$ (since $\lvert\mathbf{c}\rvert=\lvert\mathbf{a}\rvert=$ radius) **OR**	The properties of the circle should be used to show the result. Since A, B and C lies on the circle, we should note that $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=\lvert\mathbf{c}\rvert$ is the radius of circle. Also AB is the diameter of the circle, therefore, $\mathbf{b}=-\mathbf{a}$ since they are equal in length but opposite in direction.

	$\overrightarrow{AC}\cdot\overrightarrow{BC}=(\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot(-\mathbf{a})+\mathbf{a}\cdot(-\mathbf{a})$ (since $\mathbf{b}=-\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ (since $\mathbf{a}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ $=\lvert\mathbf{c}\rvert^2-\lvert\mathbf{a}\rvert^2$ (since $\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{c}\rvert^2, \mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) $=0$ (since $\lvert\mathbf{c}\rvert=\lvert\mathbf{a}\rvert=\text{radius}$) Since $\overrightarrow{AC}\cdot\overrightarrow{BC}=0$, $\therefore \angle ACB=90°$.	
(ii) **[2]**	Let points (1, 0, 0) and (0, 0, 1) be A and B respectively Let R be the point with position vector $\mathbf{r}$ Since $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$, ABR is a right-angled triangle. Therefore R lies on a sphere with AB as the diameter of the sphere.  Length of line segment joining $A(1, 0, 0)$ and $B(0, 0, 1)$ is $\sqrt{2}$ Midpoint of (1, 0, 0) and (0, 0, 1) is $\left(\frac{1}{2},0,\frac{1}{2}\right)$ which is C, the centre of the sphere. Set of vectors $\mathbf{r}$ consists of position vectors of points on a sphere with diameter $\sqrt{2}$ (OR radius $\frac{\sqrt{2}}{2}$) and centre $\left(\frac{1}{2},0,\frac{1}{2}\right)$.	The properties of the circle/sphere should be used to describe the set of vectors $\mathbf{r}$ geometrically. You should consider **(a)(i)** is related to **(a)(ii)**. Few students have managed to describe it correctly.
(b) **(i)** **[2]**	Set of vectors $\mathbf{r}$ consists of position vectors of points on a plane that contains the point M with position vector $\mathbf{m}$ and is perpendicular to the vector $\mathbf{n}$. If $\mathbf{n}$ is a unit vector, then $\lvert\mathbf{m}\cdot\mathbf{n}\rvert$ represents the shortest (perpendicular) distance from origin to the plane.  	Note that $\mathbf{m}\cdot\mathbf{n}$ is a fixed scalar value, so $\mathbf{r}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}$ defines a plane. Note that $\lvert\mathbf{m}\cdot\mathbf{n}\rvert$ is the projection of $\mathbf{m}$ onto $\mathbf{n}$ since $\mathbf{n}$ is a unit vector

(ii) [4]

Method 1

Let X, Y and Z be points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ respectively, then

$$\overrightarrow{OX}=\begin{pmatrix}x\\0\\0\end{pmatrix}, \overrightarrow{OY}=\begin{pmatrix}0\\y\\0\end{pmatrix}, \overrightarrow{OZ}=\begin{pmatrix}0\\0\\z\end{pmatrix}$$

A normal to the plane π

$$=\overrightarrow{XY}\times\overrightarrow{XZ}=\begin{pmatrix}-x\\y\\0\end{pmatrix}\times\begin{pmatrix}-x\\0\\z\end{pmatrix}=\begin{pmatrix}yz\\xz\\xy\end{pmatrix}$$

$$\therefore d=\left|\overrightarrow{OX}\cdot\frac{\begin{pmatrix}yz\\xz\\xy\end{pmatrix}}{\sqrt{y^2z^2+x^2z^2+x^2y^2}}\right|=\left|\frac{xyz}{\sqrt{y^2z^2+x^2z^2+x^2y^2}}\right|$$

$$d^2=\frac{x^2y^2z^2}{y^2z^2+x^2z^2+x^2y^2}$$

$$\frac{y^2z^2+x^2z^2+x^2y^2}{x^2y^2z^2}=\frac{1}{d^2}$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2} \quad \text{(shown)}$$

Method 1 first find the normal to the plane and make use length of projection to find d and hence the relationship

Some have also tried to find the equation of the plane and use the equation to relate to d which will yield the same result.

Method 2

Let the normal vector to the plane be $\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}$ where

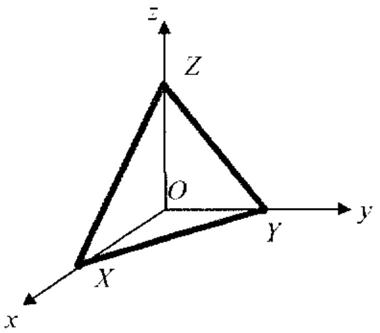
$\sqrt{n_x^2+n_y^2+n_z^2}=1$ (unit vector)

$$d=\left|\begin{pmatrix}x\\0\\0\end{pmatrix}\cdot\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}\right|\Rightarrow d=|xn_x|\Rightarrow|n_x|=\frac{d}{|x|}$$

Similarly, $d=\left|\begin{pmatrix}0\\y\\0\end{pmatrix}\cdot\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}\right|\Rightarrow|n_y|=\frac{d}{|y|}$

and, $d=\left|\begin{pmatrix}0\\0\\z\end{pmatrix}\cdot\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}\right|\Rightarrow|n_z|=\frac{d}{|z|}$

Method 2 first make use of the 3 points to find the normal to the plane which is a ***unit vector*** and then make use of the fact that the magnitude of the unit vector is 1 to find the relationship.

	$\sqrt{n_x^2+n_y^2+n_z^2}=1$ Therefore $\Rightarrow\left(\frac{d}{\lvert x\rvert}\right)^2+\left(\frac{d}{\lvert y\rvert}\right)^2+\left(\frac{d}{\lvert z\rvert}\right)^2=1$ $\Rightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2}$ (Shown) **Method 3 (Geometry)** Let X, Y and Z be points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ respectively, then  Using the base OXY, volume of the tetrahedral $OXYZ$ $=\frac{1}{3}(Area \text{ of triangle } OXY)(OZ)$ $=\frac{1}{3}\left(\frac{1}{2}[OX][OY]\right)(OZ)$ $=\frac{1}{6}\lvert xyz\rvert$ Using the base XYZ, volume of the tetrahedral $OXYZ$	**Method 3** makes use of 2 different ways to find the volume of tetrahedral using different base area and then compare these volumes to find the relationship.

$$= \frac{1}{3}(\textit{Area} \text{ of triangle } XYZ)(d)$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{XY} \times \overrightarrow{XZ}\right|\right)(d)$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\begin{pmatrix}-x\\y\\0\end{pmatrix} \times \begin{pmatrix}-x\\0\\z\end{pmatrix}\right|\right)(d), \quad \overrightarrow{XY} = \begin{pmatrix}-x\\y\\0\end{pmatrix} \text{ and } \overrightarrow{XZ} = \begin{pmatrix}-x\\0\\z\end{pmatrix}$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\begin{pmatrix}yz\\xz\\xy\end{pmatrix}\right|\right)(d)$$

$$= \frac{d}{6}\sqrt{y^2z^2 + x^2z^2 + x^2y^2}$$

Hence

$$\frac{d}{6}\sqrt{y^2z^2 + x^2z^2 + x^2y^2} = \frac{1}{6}|xyz|$$

$$\Rightarrow d^2\left(y^2z^2 + x^2z^2 + x^2y^2\right) = x^2y^2z^2$$

$$\Rightarrow \frac{y^2z^2 + x^2z^2 + x^2y^2}{x^2y^2z^2} = \frac{1}{d^2}$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{d^2}$$

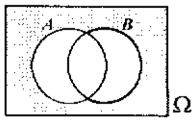
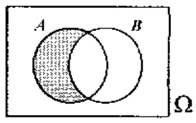
Section B: Probability and Statistics [60 marks]

5 For events A and B it is given that $\mathrm{P}(A)=0.3$, $\mathrm{P}(B|A)=0.4$ and $\mathrm{P}(A'\cap B')=0.15$. Find

(i) $\mathrm{P}(A\cup B)$, [1]

(ii) $\mathrm{P}(B)$, [3]

(iii) $\mathrm{P}(A|B')$. [2]

Solution		**Comments**
(i) **[1]**	$\mathrm{P}(A\cup B)$ $=1-\mathrm{P}(A'\cap B')$ $=1-0.15$ $=0.85$  $A'\cap B'$	Majority of students did well for this part. The most complicated looking piece of information given in the question is $\mathrm{P}(A'\cap B')$. Either you draw a venn diagram for it and then you can clearly see what you need, or you use the property that $\mathrm{P}(A'\cap B')=\mathrm{P}(A\cup B)'$
(ii) **[3]**	$\mathrm{P}(B\|A)=\dfrac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}$ $0.4=\dfrac{\mathrm{P}(B\cap A)}{0.3}$ $\mathrm{P}(B\cap A)=0.12$ $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $0.85=0.3+\mathrm{P}(B)-0.12$ $\mathrm{P}(B)=0.67$	Most student could handle this part. The main idea is to extract out $\mathrm{P}(B)$ using $\mathrm{P}(A\cup B)$ from (i) and $\mathrm{P}(A\cap B)$ which can be found from the given conditional probability. For questions which are more than 1 mark, it is imperative that students show some kind of method rather than just computation of numbers, in case the answer is wrong, they may obtain some method marks.
(iii) **[2]**	$\mathrm{P}(A\|B')$ $=\dfrac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}$ $=\dfrac{\mathrm{P}(A)-\mathrm{P}(A\cap B)}{1-\mathrm{P}(B)}$ $=\dfrac{0.3-0.12}{1-0.67}$ $=\dfrac{6}{11}$  $A\cap B'$	This part was generally well done, except for carryover of wrong answer from (ii), or, having obtained 0.67 for (ii) but forgetting to take the complement for the denominator. It is heartening to note most students could recognize the property used in the numerator, from section 2.2 (vi) in the lecture notes. Alternatively, drawing a venn diagram would help.

		This is a less popular method, with only a handful of students who tried with this way. Nonetheless, students using either method should familiarize themselves with the concept behind the other method. The tree diagram method demonstrates a clear understanding of where the conditional probability belongs to in the tree. Note that with this method, the answer for (ii) will be obtained first.
(i) [1]	$\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)$ $=0.3+0.67-0.12$ $=0.85$	
(iii) [2]	$\mathrm{P}(A \mid B')$ $=\dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')}$ $=\dfrac{0.18}{1-0.67}$ $=\dfrac{6}{11}$	

6 The recruitment manager of the private car hire company, I-ber, claims that the mean weekly earnings of a full-time driver is \$980. The managing director suspects that the mean weekly earnings is less than \$980 and he instructs the recruitment manager to carry out a hypothesis test on a sample of drivers. It is given that the population standard deviation of the weekly earnings is \$88.

(i) State suitable hypotheses for the test, defining any symbols that you use. [2]

The recruitment manager takes a random sample of 10 drivers. He finds that the weekly earnings in dollars, are as follows.

942 950 905 1003 883 987 924 920 913 968

(ii) Find the mean weekly earnings of the sample of these 10 drivers. Carry out the test, at 5% level of significance, for the recruitment manager. Give your conclusion in context and state a necessary assumption for the test to be valid. [5]

(iii) Find the smallest level of significance at which the test would result in rejection of the null hypothesis, giving your answer correct to 1 decimal place. [1]

Solution		**Comments**
(i) **[2]**	Null hypothesis, $H_0 : \mu = 980$ Alternative hypothesis, $H_1 : \mu < 980$ where μ is the population mean weekly salary.	This is a similar question to CT, the word "population" should be there.
(ii) **[5]**	Let X be the weekly earning of an I-ber driver (in \$). Using GC, $\bar{x} = 939.5$ Perform a 1-tailed test at 5% significance level. Under H_0, $\bar{X} \sim \mathrm{N}\left(980, \frac{88^2}{10}\right)$ p-value $= 0.0728 > 0.05$, hence we do not reject H_0, and conclude that, based on the test carried out by the recruitment manager, there is insufficient evidence for the managing director to conclude at 5% level of significance that the mean weekly earnings of a driver is less than \$980. Assumption: Assume that the weekly earnings of the I-ber drivers are normally distributed.	Defining X should be a default first step, if it is not defined in the question. **Please follow the school prescribed presentations provided in various softcopies (including this one) to avoid losing marks unnecessarily.** The conclusion was sometimes not done properly, for example saying "it is not \$980" instead of "less than \$980"

		There was also some confusion between the claims of the recruitment manager vs the managing director. Finally do also take note that we either reject H_0 (and thus accept H_1) or do not reject H_0. We **do not** conclude that we accept H_0
(iii) [1]	To reject H_0, smallest level of significance = 7.3% (1 d.p)	Students performed below expectations for this part. It should be noted that the level of significance is usually given in percentage form and thus it should be clear how to proceed.

7 **In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A company sells hand sanitiser in bottles of two sizes – small and large. The amounts, in ml, of hand sanitiser in the small and large bottles, are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard deviation
Small bottles	108	5
Large bottles	510	σ

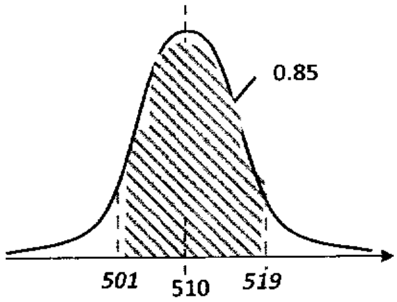
(i) Find the probability that the amount of hand sanitiser in a randomly chosen small bottle is less than 100 ml. [1]

(ii) During a quality control check on a batch of small bottles of hand sanitiser, 100 small bottles are randomly chosen to be inspected by an officer one at a time. Once he finds five bottles, each with amount of hand sanitiser less than 100 ml, that batch will be rejected. Find the probability that he had to check through all 100 bottles to reject that batch. [2]

(iii) Given that the amount of hand sanitiser in 85% of the large bottles lie within 9 ml of the mean, find σ. [3]

(iv) Given instead that $\sigma = 6$, find the probability that the amount of hand sanitiser in a randomly chosen large bottle is less than five times the amount of hand sanitiser in a randomly chosen small bottle. [3]

Solution		**Comments**
(i) [1]	Let X and Y be the amount, in ml, of hand sanitiser in a small and large bottle respectively. Then $X \sim \mathrm{N}(108, 5^2)$ and $Y \sim \mathrm{N}(510, \sigma^2)$ $\mathrm{P}(X < 100) = 0.054799 = 0.0548$ (3 s.f.)	Majority did this part well, but some still keyed in wrongly as normalcdf(-E99,100, 108, **25**) instead of normalcdf(-E99,100, 108, **5**)
(ii) [2]	Required probability $= {}^{99}C_4 (0.054799)^4 (1-0.054799)^{95} (0.054799)$ $= 0.00880$ (3 s.f.)	There are 4 bottles from the first 99 with the criteria of less than 100ml, and the 5th is the 100th being examined, so some forgot to multiply by 0.054799 for this last bottle. Remember to use 5.s.f and to round your answers properly.
(iii) [3]	$\mathrm{P}(\lvert Y-510\rvert < 9) = 0.85$ $\mathrm{P}\left(\frac{\lvert Y-510\rvert}{\sigma} < \frac{9}{\sigma}\right) = 0.85$ $\mathrm{P}\left(\lvert Z\rvert < \frac{9}{\sigma}\right) = 0.85$	Majority did this part well, but there were some who mistakenly thought that it is from 505.5 to 514.5 (an interval of 9 ml) which is a misinterpretation of "within 9ml of the mean".

	From G.C, $\frac{9}{\sigma}=1.4395$ $\sigma=6.25$ (3 s.f.) **OR**  $\mathrm{P}(501<Y<519)=0.85$ $\mathrm{P}\left(\frac{501-510}{\sigma}<\frac{Y-510}{\sigma}<\frac{519-510}{\sigma}\right)=0.85$ $\mathrm{P}\left(-\frac{9}{\sigma}<Z<\frac{9}{\sigma}\right)=0.85$ From G.C, $\frac{9}{\sigma}=1.4395$ $\sigma=6.25$ (3 s.f.)	Some also thought that the standardization is $\frac{Y-510}{\sigma^2}$, where it should be σ, and not σ^2. Some used invNorm(0.85,0,1, LEFT) instead of invNorm(0.85,0,1, CENTER) which led to a wrong value. Some who did $\mathrm{P}(Y<501)=\frac{1-0.85}{2}=0.075$ were able to successfully get the answer. Those who tried a graphical method by plotting Y_1= normalcdf(501,519,510,X) and Y_2=0.85 were mostly successful. Others who mistakenly thought that σ is an integer, used a tabular method, and were not given full credit as this answer is not to 3.s.f.
(iv) **[3]**	$Y-5X \sim \mathrm{N}\left(510-5(108), 6^2+5^2\left(5^2\right)\right)$ i.e $Y-5X \sim \mathrm{N}(-30, 661)$ $\mathrm{P}(Y-5X<0)=0.878$ (3 s.f.)	Majority did well on this, but quite some made calculation mistakes for the $\operatorname{Var}(Y-5X)$: swopped the standard deviations for X and Y, forgot the square for the scaling factor 5, used subtraction instead of addition, etc. Some who successfully obtained the correct distribution also made the unfortunate error: used normalcdf(-E99, 0,-30,661) instead of $\sqrt{661}$.

8 This question is about arrangements of all eight letters in the word IMMUNITY.

(i) Show that the number of different arrangements of the eight letters that can be made is 10080. [1]

(ii) Find the number of different arrangements that can be made with no two vowels next to each other. [3]

One of the 10080 arrangements in part **(i)** is randomly chosen.
Let A denote the event that the two I's are next to each other and let B denote the event that the two M's are next to each other.

(iii) Determine, with a reason, whether A and B are
(a) mutually exclusive, [1]

(b) independent. [3]

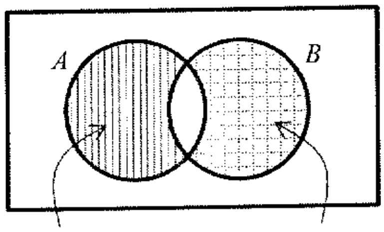
(iv) Find the probability that the chosen arrangement contains no two adjacent letters that are the same. [4]

	Solution	Comments
(i) [1]	The number of different arrangements $=\dfrac{8!}{2!2!}=10080$ (shown)	This part was expectedly well-done.
(ii) [3]	The consonants are M, M, N, T, Y and the vowels are I, I, U. Number of ways to arrange the consonants $=\dfrac{5!}{2!}=60$ $C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5$ $\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$ Number of ways to choose 3 slots to insert the vowels $={}^6C_3=20$ Number of ways to arrange the vowels $=\dfrac{3!}{2!}=3$ $\therefore$ The number of different arrangements $=60\times 20\times 3=3600$ **<u>Method 2</u>** (complementary approach) : Case 1 : 3 vowels are together (as a unit) [I I U] M M N T Y Number of arrangements $=\dfrac{6!}{2!}\times 3=1080$.	You must know what are vowels (or consonants). Some candidates were confused about these. We recommend the first approach. Methods 2 and 3 are for your reference.

	Case 2 : Group 2 'I's together (as a unit) M M N T Y [I I] U Number of arrangements = $\frac{7!}{2!}$ = 2520. Case 3 : Group one I and one U together (as a unit) M M N T Y [I U] I Number of arrangements = $\frac{7!}{2!}\times 2 = 5040$. Hence, the required number of different arrangements = 10080 − 2520 − 5040 + 1080 = 3600. **<u>Method 3</u>** (complementary approach) : Case 1 : 3 vowels are together (as a unit) [I I U] M M N T Y Number of arrangements = $\frac{6!}{2!}\times 3$ = 1080. Case 2 : Group 2 'I's together (as a unit), and U is another unit not adjacent to I-I [I I] [U] M M N T Y ↑ ↑ ↑ ↑ ↑ ↑ Number of ways to arrange the 5 consonants = $\frac{5!}{2!}$ = 60. Numbers of ways to insert 2 units = ${}^{6}C_{2}\times 2 = 30$. Number of different arrangements = 60 × 30 = 1800.	A gentle reminder that it is your responsibility to briefly describe the different cases. For Method 2, do note that the 3 vowels being together also happens in case 2 and case 3. For Method 3, do note that in case 2 and case 3, special care is taken to make sure the 3 vowels don't come together.

	Case 3 : Group one I and U together (as a unit), and the other I is another unit not adjacent to I-U unit [I U] [I] M M N T Y ↑ ↑ ↑ ↑ ↑ ↑ Number of ways to arrange the 5 consonants = $\frac{5!}{2!}=60$. Numbers of ways to insert 2 units = ${}^{6}C_2 \times 2 \times 2 = 60$. Number of different arrangements = $60 \times 60 = 3600$. Hence, the required number of different arrangements $= 10080 - 1080 - 1800 - 3600$ $= 3600$.	
(iii) (a) [1]	Events A and B are not mutually exclusive. This is because both events can occur at the same time, for example arrangements such as MMUNIITY, IINTUMMY and YUMMIINT.	The most direct approach is to give an explicit example of the existence of such an arrangement. Another viable approach is to find $\mathrm{P}(A \cap B)$ and show that it is non-zero.
(iii) (b) [3]	$\mathrm{P}(A)=\mathrm{P}(B)=\frac{7!}{2!}\div 10080=\frac{1}{4}$, so $\mathrm{P}(A)\times\mathrm{P}(B)=\frac{1}{16}$ $\mathrm{P}(A \cap B)=\frac{6!}{10080}=\frac{1}{14}$ $\mathrm{P}(A \cap B) \neq \mathrm{P}(A)\times\mathrm{P}(B) \Rightarrow$ Events A and B are not independent.	You **shouldn't** be trying to explain your way through words here. You **ought to be** making attempt to work out actual probabilities to establish one of the following : $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$ $\mathrm{P}(A \mid B)=\mathrm{P}(A)$ $\mathrm{P}(B \mid A)=\mathrm{P}(B)$
(iv) [4]	Required probability $=\mathrm{P}(A' \cap B')$ $=1-\mathrm{P}(A \cup B)$ $=1-[\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)]$ $=1-\left[\frac{1}{4}+\frac{1}{4}-\frac{1}{14}\right]$ $=\frac{4}{7}$	There are plenty instances of candidates getting "carried away" and find number of arrangements instead of probabilities.

Method 2 (Venn Diagram) :



$P(A)-P(A\cap B)=\frac{1}{4}-\frac{1}{14}=\frac{5}{28}$ $P(B)-P(A\cap B)=\frac{1}{4}-\frac{1}{14}=\frac{5}{28}$

Required probability $= 1-\frac{5}{28}-\frac{5}{28}-\frac{1}{14}=\frac{4}{7}$.

Method 2. Visualizing with the aid of Venn Diagram can be useful.

Method 3 (complementary approach) :

Case 1 : Group 2 'M's together (as a unit), and the two 'I's are to be separated from each other.

I I

↑ [MM] ↑ U ↑ N ↑ T ↑ Y ↑

Number of ways $= 5! \times {}^6C_2 = 1800$.

Case 2 : Group 2 'I's together (as a unit), and the two 'M's are to be separated from each other.

M M

↑ [II] ↑ U ↑ N ↑ T ↑ Y ↑

Number of ways $= 5! \times {}^6C_2 = 1800$.

Case 3 : Group 2 'I's together (as a unit), and group 2 'M's together (as a unit)

[II] [MM] U N T Y

Number of ways $= 6! = 720$.

Hence, the required probability $= 1-\frac{1800}{10080}-\frac{1800}{10080}-\frac{720}{10080}=\frac{4}{7}$.

Methods 3 and 4 are comparable ideas. However, you ought to be careful with the "overlap" in Method 4.

	Method 4 (complementary approach) : Case 1 : Group 2 'M's together (as a unit) [MM] U N T Y I I Number of ways $= \dfrac{7!}{2!} = 2520$. Case 2 : Group 2 'I's together (as a unit) [II] U N T Y M M Number of ways $= \dfrac{7!}{2!} = 2520$. Case 3 : Group 2 'I's together (as a unit), and group 2 'M's together (as a unit) [II] [MM] U N T Y Number of ways $= 6! = 720$. Hence, the required probability $= 1 - \dfrac{2520}{10080} - \dfrac{2520}{10080} + \dfrac{720}{10080} = \dfrac{4}{7}$.	Do note that in case 1 and case 2, there are instances of I-I grouping together, and M-M grouping together.

9 A bag initially contains 3 red balls and 3 black balls. Whenever a red ball is drawn from the bag, it is put back into the bag together with an extra red ball. Whenever a black ball is drawn from the bag, it is not put back into the bag and no extra balls are added.

Isaac draws n balls from the bag, one after another, where $n \in \mathbb{Z}^+$, and R denotes the number of red balls out of the n balls drawn.

(a) Give two reasons why R cannot be modelled using a Binomial distribution. [2]

(b) For $n = 3$, find

(i) $\mathrm{P}(R \geq 1)$, [2]

(ii) the probability that the first ball drawn is black given that at least 1 of the 3 balls drawn is red. [3]

(c) For $n = 31$, show that $\mathrm{P}(R = 31) = \dfrac{1}{714}$. [2]

(d) Isaac wins 100 dollars for each red ball he draws if all the balls he draws from the bag are red, and does not win any money otherwise. What is the maximum amount of money Isaac would win if the probability of all the balls he draws are red exceeds 0.0001? [3]

Solution		Comments
(a) **[2]**	1. The probability of drawing red balls from 2nd draw does not remain at a constant. If first ball is red, probability of red for 2nd ball is 4/7 If first ball is black, probability of red for 2nd ball is 3/6. 2. The drawing of balls are not independent of each other as it involves replacement.	Remember to explain in context of the question during A level.
(b) **(i)** **[2]**	$\mathrm{P}(R \geq 1) = \mathrm{P}(\text{at least 1 red}) = 1 - \mathrm{P}(\text{all black})$ $= 1 - \left(\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)$ $= 1 - \frac{1}{20} = \frac{19}{20}$	The most efficient way is to use complement in this case. Those using listing need to evaluate the probability of every term: P(RRR), P(RRB), P(RBR), P(BRR), P(RBB), P(BRB), P(BBR).
(ii) **[3]**	Let A be the event the first ball drawn is black. Then	Some students didn't realise that "at least one of the 3 balls is

	$\text{P}(A \mid R \geq 1)$ $= \dfrac{\text{P}(A \cap \{R \geq 1\})}{\text{P}(R \geq 1)}$ $= \dfrac{\text{P}(\{B,R\}+\{B,B,R\})}{\frac{19}{20}}$ $= \dfrac{\frac{1}{2}\left(\frac{3}{5}\right)+\frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{3}{4}\right)}{\frac{19}{20}}$ $= \dfrac{9}{19}$ Alternatively, $\text{P}(A \mid R \geq 1)$ $= \dfrac{\text{P}(A \cap \{R \geq 1\})}{\text{P}(R \geq 1)}$ $= \dfrac{\text{P}(B)-P(BBB)}{\frac{19}{20}}$ $= \dfrac{\frac{1}{2}-\frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)}{\frac{19}{20}}$ $= \dfrac{9}{19}$	red" refers to "$R \geq 1$", and calculate P(at least 1 of the 3 balls is red). For numerator, P(BR) = P(BRR) + P(BRB) Note that P(BBR)≠P(BRB)
(c) [2]	P(all 31 balls red) $= \left(\frac{3}{6}\right)\left(\frac{4}{7}\right)\left(\frac{5}{8}\right)\left(\frac{6}{9}\right)\left(\frac{7}{10}\right)\cdots\left(\frac{30}{33}\right)\left(\frac{31}{34}\right)\left(\frac{32}{35}\right)\left(\frac{33}{36}\right)$ $= \dfrac{3(4)(5)}{34(35)(36)}$ $= \dfrac{1}{34(7)(3)}$ $= \dfrac{1}{714}$	The idea here is to list down sufficient terms to observe that (6)(7)(8)…(32)(33) appears in both the numerator and denominator.

(d) [3]

P(all red balls in first n draws) $= \mathrm{P}(R = n)$

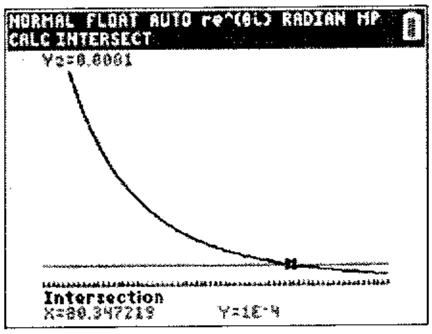
$= \left(\frac{3}{6}\right)\left(\frac{4}{7}\right)\left(\frac{5}{8}\right)\left(\frac{6}{9}\right)\cdots\left(\frac{n}{n+3}\right)\left(\frac{n+1}{n+4}\right)\left(\frac{n+2}{n+5}\right)$

$= \frac{60}{(n+3)(n+4)(n+5)}$

From GC,

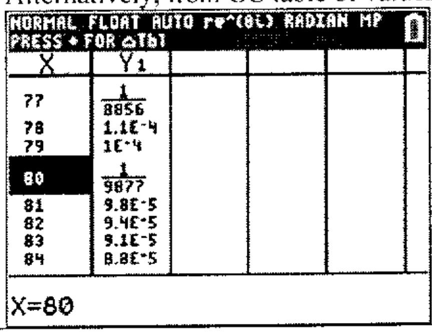
$\frac{60}{(n+3)(n+4)(n+5)} > 0.0001$

$\Rightarrow 0 < n < 80.347$



Maximum amount Isaac would win with probability exceeding 0.0001 is \$8000.

Alternatively, from GC table of values:



NORMAL FLOAT AUTO re^(θi) RADIAN MP
PRESS + FOR ΔTbl

X	Y1
77	1/8856
78	1.1E-4
79	1E-4
80	1/9877
81	9.8E-5
82	9.4E-5
83	9.1E-5
84	8.8E-5

X=80

Comments:

The idea is to see that $\mathrm{P}(R = n) > 0.0001$ where part (c) provides the hint on getting the general form for $\mathrm{P}(R = n)$.

GC does the rest of the calculations. Some students managed to get the general form but gotten different answers from their GCs, most likely due to carelessness.

10 A bag contains four balls numbered 1, 2, 3 and 4. In a game, a ball is drawn at random from the bag and then a fair coin is tossed a number of times that is equal to the number shown on the ball drawn. The random variable X is the number of heads recorded.

(i) Show that $\mathrm{P}(X=0)=\frac{15}{64}$. Find $\mathrm{P}(X=x)$ for all other possible values of x. [5]

(ii) Denoting the expectation and variance of X by μ and σ^2 respectively, find $\mathrm{P}(X>\mu)$ and show that $\sigma^2=\frac{15}{16}$. [3]

Adam plays this game 10 times.

(iii) Find the probability that there are at least two games with at least 2 heads recorded. [2]

Bill plays this game 50 times.

(iv) Using a suitable approximation, estimate the probability that the average number of heads recorded is less than 1. [3]

Solution		**Comments**
(i) [5]	The following table * shows the probabilities of obtaining the number of heads corresponding to the number of throws.	

No. of tosses (No. shown on the ball drawn)	Number of heads recorded, X: 0	1	2	3	4
1	T: $\frac{1}{4}\times\binom{1}{0}\left(\frac{1}{2}\right)$	H: $\frac{1}{4}\times\binom{1}{1}\left(\frac{1}{2}\right)$	-	-	-
2	TT: $\frac{1}{4}\times\binom{2}{0}\left(\frac{1}{2}\right)^2$	HT or TH: $\frac{1}{4}\times\binom{2}{1}\left(\frac{1}{2}\right)^2$	HH: $\frac{1}{4}\times\binom{2}{2}\left(\frac{1}{2}\right)^2$	-	-
3	TTT: $\frac{1}{4}\times\binom{3}{0}\left(\frac{1}{2}\right)^3$	HTT or THT or HTT: $\frac{1}{4}\times\binom{3}{1}\left(\frac{1}{2}\right)^3$	HHT or HTH or THH: $\frac{1}{4}\times\binom{3}{2}\left(\frac{1}{2}\right)^3$	HHH: $\frac{1}{4}\times\binom{3}{3}\left(\frac{1}{2}\right)^3$	-
4	TTTT: $\frac{1}{4}\times\binom{4}{0}\left(\frac{1}{2}\right)^4$	HTTT, THTT, TTHT or TTTH: $\frac{1}{4}\times\binom{4}{1}\left(\frac{1}{2}\right)^4$	HHTT, HTHT, HTTH, THHT, THTH or TTHH: $\frac{1}{4}\times\binom{4}{2}\left(\frac{1}{2}\right)^4$	HHHT, HHTH, HTHH or THHH: $\frac{1}{4}\times\binom{4}{3}\left(\frac{1}{2}\right)^4$	HHHH: $\frac{1}{4}\times\binom{4}{4}\left(\frac{1}{2}\right)^4$

	$P(X=0)=P(\text{T, TT, TTT or TTTT})=\frac{1}{4}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right]=\frac{15}{64}$ (shown)	
	<table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>$P(X=x)$</td><td>$\frac{15}{64}$</td><td>$\frac{13}{32}$</td><td>$\frac{1}{4}$</td><td>$\frac{3}{32}$</td><td>$\frac{1}{64}$</td></tr></table>	$P(X=x)$ can be obtained by adding the respective probabilities in the corresponding column of table *. Do check that $\sum_{x=0}^{4} P(X=x)=1$.
(ii) [3]	$\mu=E(X)$ $=\left(0\times\frac{15}{64}\right)+\left(1\times\frac{13}{32}\right)+\left(2\times\frac{1}{4}\right)+\left(3\times\frac{3}{32}\right)+\left(4\times\frac{1}{64}\right)$ $=\frac{5}{4}=1.25$ $P(X>\mu)=P\left(X>\frac{5}{4}\right)$ $=P(X=2)+P(X=3)+P(X=4)$ $=\frac{1}{4}+\frac{3}{32}+\frac{1}{64}$ $=\frac{23}{64}$ or 0.359375 $E(X^2)=\left(0^2\times\frac{15}{64}\right)+\left(1^2\times\frac{13}{32}\right)+\left(2^2\times\frac{1}{4}\right)+\left(3^2\times\frac{3}{32}\right)+\left(4^2\times\frac{1}{64}\right)=\frac{5}{2}$ $\sigma^2=\text{Var}(X)$ $=E(X^2)-[E(X)]^2$ $=\frac{5}{2}-\left(\frac{5}{4}\right)^2$ $=\frac{15}{16}$ (shown)	X is a discrete random variable. Many mistook X to be normally distributed, and ***wrongly concluded*** that $P(X>\mu)=\frac{1}{2}$.
(iii) [2]	Let Y be the number of games, out of ten, with at least two heads. $Y\sim B\left(10,\frac{23}{64}\right)$	$P(X\geq 2)=P(X>\mu)$, whose value has been found in (ii)

	$P(Y \geq 2)$ $= 1 - P(Y \leq 1)$ $= 1 - 0.076953$ $= 0.923$ (3 s.f.)	
(iv) **[3]**	Let $\overline{X}$ be the average number of heads recorded in 50 games. Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim N\left(1.25, \frac{15}{16(50)}\right)$ or $\overline{X} \sim N\left(1.25, \frac{0.9375}{50}\right)$ approximately. $P\left(\overline{X} < 1\right) = 0.0339$ (3 s.f.) Alternatively, Since $n = 50$ is large, by Central Limit Theorem, $X_1 + X_2 + \cdots X_{50} \sim N\left(50 \times 1.25, 50 \times \frac{15}{16}\right)$ approximately. $P\left(\overline{X} < 1\right) = P\left(X_1 + X_2 + \cdots X_{50} < 50\right) = 0.0339$ (3 s.f.)	It is ***wrong*** to state that $\overline{X} \sim N\left(1.25, \frac{15}{16}\right)$ due to Central Limit Theorem. Note that $\overline{X} = \frac{X_1 + X_2 + \cdots X_{50}}{50}$