16

RIVER VALLEY HIGH SCHOOL
2021 JC2 Preliminary Examination
Higher 2

NAME	

CLASS					

INDEX NUMBER	

MATHEMATICS

Paper 1

9758/01

16 Sep 2021

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

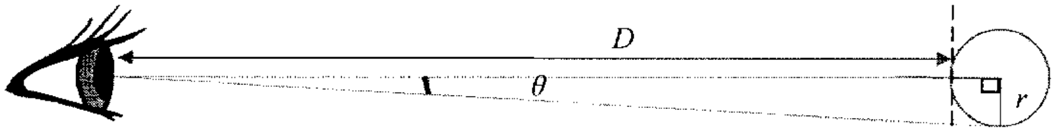
For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total	

Calculator Model:

This document consists of ___ printed pages and ___ blank pages.

1 It is given that $y = f(x)$ is a function such that $f(0) = 1$ and $\frac{dy}{dx} = 1 + y^2$. Find the first three non-zero terms in the series expansion of $\frac{f(x)}{\sqrt{9-x}}$. [4]

2 The *angular displacement* of an object is how wide that object appears to be from an observer's point of view.



The angular displacement θ, of an object with radius r km that is D km away is shown in the diagram above.

Oumuamua is the first interstellar object observed in our solar system. Discovered in 2017, it had an angular displacement of 2.85×10^{-10} radians. Powerful Earth-based lasers measured its distance as 3.59×10^{9} km from Earth.

(i) Show that the radius of *Oumuamua* is approximately 1.023 km, rounded to 3 decimal places. [2]

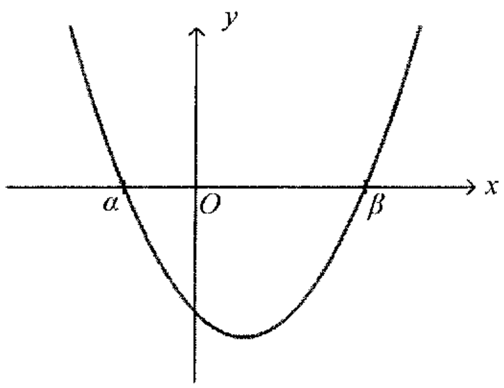
(ii) The rate of change of angular displacement is found to be -1.095×10^{-14} radians per hour at this instant. Assuming θ is sufficiently small for θ^3 and higher powers to be ignored, find the rate at which *Oumuamua* is moving away from Earth. [4]

3 It is given that $1-i$ is a root of the equation $2z^4 + pz^3 + 8z^2 + qz + 4 = 0$, where p and q are real constants.

(i) Write down $(1-i)^2$, $(1-i)^3$ and $(1-i)^4$ in cartesian form. Hence find the values of p and q. [3]

(ii) Without the use of calculator, find the other roots of the equation in exact form. [4]

4



The diagram shows the graph of $y = x^2 - 2x - 5$. The two roots of the equation $x^2 - 2x - 5 = 0$ are denoted by α and β, where $\alpha < \beta$.

(i) Find the exact values of α and β. [2]

A sequence of positive numbers $x_1, x_2, x_3, \ldots$ is such that

$$x_{n+1} = (2x_n + 5)^{\frac{1}{2}} \text{ for } n = 1, 2, 3, \ldots.$$

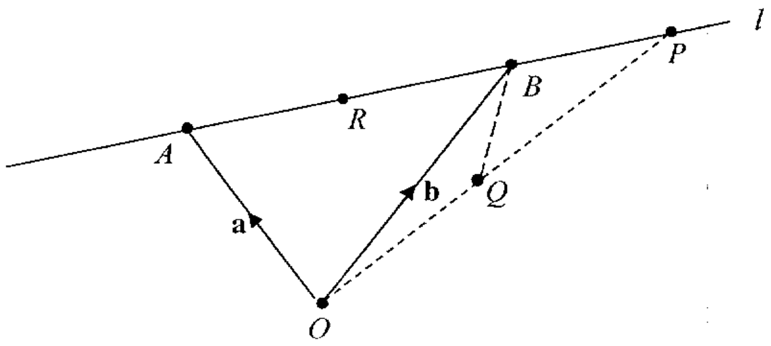
As $n \to \infty$, $x_n \to l$.

(ii) Explain why l satisfies the equation $l^2 - 2l - 5 = 0$. [1]

(iii) Show that $l = \beta$. [2]

(iv) By considering the graph of $y = x^2 - 2x - 5$ and the sign of y, show that when $0 < x_n < \beta$, then $x_n < x_{n+1}$. [2]

5 The diagram below shows a straight line l passing through the points A and B. With reference to the origin O, the position vectors of A and B are **a** and **b** respectively. It is further given that **a** is a unit vector, $|\mathbf{b}| = 2$ and $\angle AOB = 60°$.



(i) State the values of $\mathbf{a}\cdot\mathbf{b}$ and $|\mathbf{a}\times\mathbf{b}|$. [2]

(ii) The point P lies on the line l and is such that $AB : BP = 2:1$. The point Q on the line OP is such that $\overrightarrow{OQ} = \lambda\overrightarrow{OP}$ where $0 < \lambda < 1$. Determine the value of λ such that the area of triangle OBQ is $\frac{1}{2\sqrt{3}}$ of the area of triangle OAB. [3]

(iii) It is further given that the point R on the line l is such that $\angle AOR = \angle ROB$. Show that R has position vector $\mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$ for some $\mu \in \mathbb{R}$ and hence find this value of μ. [3]

6 Aeroplanes need to maintain a certain speed in order to generate enough lift to stay aloft. However, moving quickly is inefficient at low altitudes, where there is greater air resistance. There is less air resistance at higher altitudes, but planes need to move faster in order to stay aloft as there is less air.

The following formula relates the speed of a particular aeroplane necessary to stay aloft to the altitude at which it is travelling:

$$v = 0.8h^3 - 18h^2 + 170h,$$

where v is the minimum speed in kilometres per hour, to stay aloft at an altitude of h thousand kilometres.

It is also given that

$$E = \frac{1000(700)^2}{700^2 + (v - 700)^2},$$

where E is the fuel efficiency of a suitable unit, at which the aeroplane is travelling.

(i) Show that $\frac{dv}{dh} > 0$ for all h. [2]

(ii) By differentiation, find the height at which the aeroplane should travel in order to achieve the greatest fuel efficiency. [5]

7 An insurance company, Super Western, launched a new investment plan. The investment plan offers a fixed interest rate of 5% of the amount available in the plan at the start of that year. The interest is added to the plan at the end of each year. Tom and Mary decide to invest in this plan.

(a) Mary decides to place \$$y$ at the start of the first year and then a further \$$y$ again at the start of each subsequent year. She chooses to leave the money in the investment plan and let the interest accumulate.

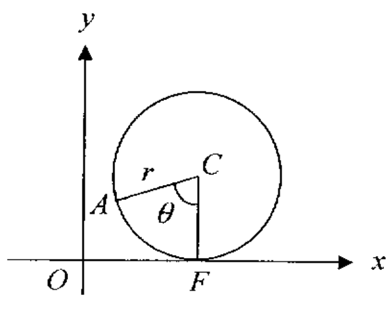
(i) How much money will there be in the investment plan at the end of 1 year? [1]

(ii) Suppose that the interest of the final year has been added into the plan, show that at the end of n years, Mary will have a total amount of $\$21(1.05^n - 1)y$ in her plan. [3]

(iii) Calculate the number of complete years it takes Mary to have at least $\$15y$ in her plan. [2]

(b) Tom decides to utilize the investment plan differently. He plans to withdraw the interest immediately when the interest is being added into his plan. Suppose that Tom invests \$$3x$ at the start of first year and \$$2x$ at start of each subsequent year, what is the total amount of interest he has withdrawn at the end of n years? [3]

8 A wheel with centre C is pushed along a flat surface in a straight line. The point A on the wheel, is initially in contact with the ground at O. After the wheel has rotated through an angle of θ radians, the point of contact with the ground is F and the length of the arc AF is equal to OF.



The wheel has a fixed radius of r.

(a) Show that the coordinates of A after the wheel has rotated through an angle of θ radians is

$$\left(r\theta - r\sin\theta,\ r - r\cos\theta\right).$$

Hence or otherwise, find the cartesian equation of the locus of A as the wheel is pushed along the surface, for $0 \leq \theta \leq 2\pi$. Express your answer as x in terms of y and r. [4]

(b) The parametric equation of the curve C is given by

$$x = rt - r\sin t$$

$$y = r - r\cos t,$$

for $0 \leq t \leq 2\pi$, where r is a constant.

(i) Show that $\int y \, dx = r^2 \int (1-\cos t)^2 \, dt$. [2]

(ii) Hence or otherwise, find the exact area of the region bounded by C and the x-axis. [3]

9 The curve C has equation $y = \dfrac{x^2 + x - 2}{x+k}$, where $x \in \mathbb{R}$, $x \neq -k$ for some $k \in \mathbb{R}$.

(i) Find the range of values of k such that C has 2 stationary points. [3]

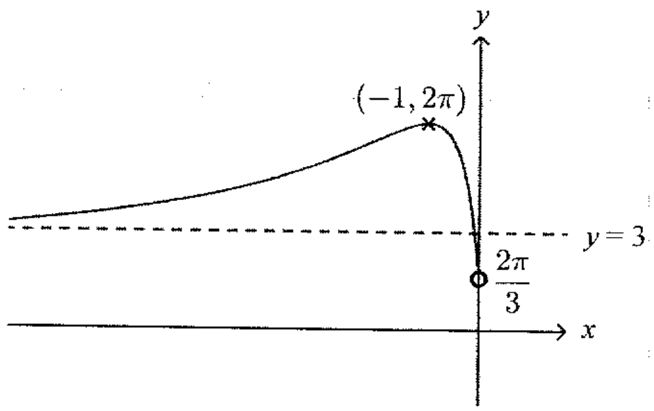
(ii) Sketch C for $k = 3$, stating the equation of any asymptotes, the coordinates of stationary points and points where the curve crosses the axes. [3]

(iii) By adding a suitable graph to the graph in part (ii), solve the inequality $x^4 + x^3 - 2x^2 - x - 3 \leq 0$ for values of $x > -3$. [3]

10 The function f is defined by $f : x \mapsto \sin x + x$, for $x \in \mathbb{R}, x > 0$.

(i) Show that $f'(x) \geq 0$. Hence, or otherwise, show that f^{-1} exists. [3]

(ii) Find the values of x for which $f(x) = f^{-1}(x)$. [2]



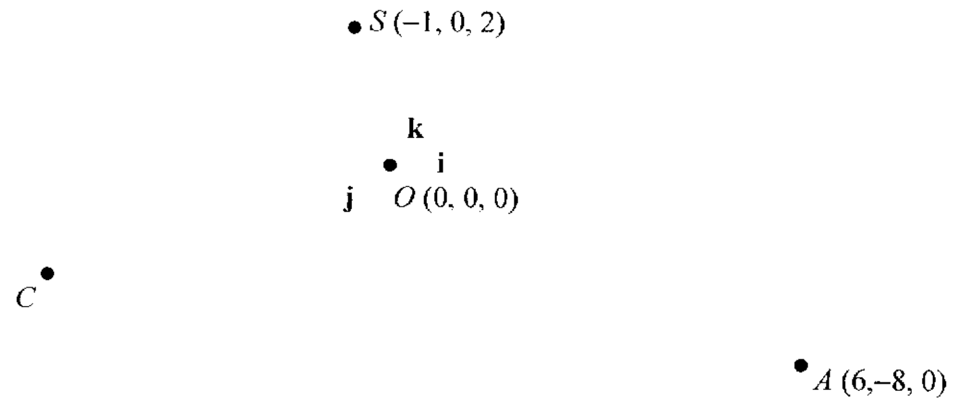
The diagram above shows the graph of $y = g(x)$.

(iii) Show that the composite function fg exists. Hence, determine the range of fg. [4]

(iv) The function h is defined by $h(x) = g(x)$ for $x \leq -1$. State the value of $h^{-1}f(2\pi)$. [1]

11 In an indoor playground, a virtual reality enclosure is being set up. The base of the enclosure takes on the shape of a triangle OCA as shown in the diagram. The point O represents the origin and the point A has coordinates $(6,-8, 0)$. The highest point of the enclosure is at the point S with coordinates $(-1, 0, 2)$. The walls of the enclosure are represented by triangles SCA, SCO and SAO respectively.

The line that passes through C and A has equation $\frac{6-x}{2} = y+8,\ z=0$. The base of the enclosure is represented by the plane with equation $z=0$.



(i) Find the cartesian equation of the plane representing the wall SCA. Hence, find the acute angle between the base of the enclosure and the wall SCA. [4]

(ii) A laser projector is to be set up at a point F along the line segment CA such that it is closest to the point S. Find the coordinates of F. [3]

(iii) The projector at F emits a laser beam that makes a point on wall SAO to create the illusion of a shooting star. The projected point moves in a straight line on the wall SAO from S to the point N which is closest to F. Find the length of FN. [3]

(iv) Calculate the angle that the laser beam emitted from the projector sweeps through as the point moves from S to N. [2]

12 (i) The drag force experienced by an object moving through a medium can be modelled by the differential equation

$$\frac{dv}{dt} = -\mu v - \lambda v^3,$$

where v is the velocity of the object in metres per second, t is time in seconds, and μ and λ are real positive constants.

Using the substitution $z = \frac{1}{v^2}$, show that $\frac{dz}{dt} = 2\mu z + 2\lambda$.

Hence, find the general solution for $v > 0$, in terms of t. [8]

(ii) State the long term behaviour of v. [1]

For the rest of the question, take $\mu = 0.1$ and $\lambda = 0.08$.

(iii) Find the particular solution of v for which the object is travelling at 1 ms^{-1} after 3 s. [3]

END OF PAPER

RVHS 2021 H2 MA Prelim Paper 2

Section A: Pure Mathematics [40 marks]

1 The function f is defined by $f(z) = az^3 + bz^2 + cz + d$ where a, b, c and d are real numbers. Given that $1-\sqrt{3}\,\mathrm{i}$ and $\frac{3}{2}$ are roots of $f(z)=0$, find b, c and d in terms of a. [4]

2 (i) By using the formulae for $\sin(A \pm B)$, show that

$$\sin\left(p+\frac{1}{2}\right)\theta - \sin\left(p-\frac{1}{2}\right)\theta \equiv 2\cos p\theta \sin\frac{1}{2}\theta.$$ [1]

(ii) Hence show that $\sum_{p=1}^{n} \cos p\theta = \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\left[\sin\left(n+\frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta\right]$. [2]

(iii) Using the result in part (ii), find the exact value for $\sum_{p=5}^{32} \cos(p-2)\theta$ when $\theta = \frac{\pi}{2}$. [3]

3 A curve C has equation $y^2 + xy = -1$, where $y > -1$.

(a) Describe a sequence of transformations that will transform the graph of C onto the graph of $4y^2 - 2(x-1)y = -1$. [3]

(b) (i) Show that the gradient function of C can be expressed as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{2y+x}$. [2]

(ii) For a non-zero constant a, find the equation of the tangent to the point where $y = a$ in terms of a. [3]

(iii) State the behaviour of the tangent as $a \to 1$. [1]

4 (i) Verify that $8e^{\mathrm{i}\left(2k\pi - \frac{\pi}{2}\right)} = -8\mathrm{i}$, where k is an integer. [2]

(ii) Given that $\theta = \frac{2k\pi}{3} - \frac{\pi}{6}$ where k is an integer and $-\pi < \theta \le \pi$, show that $k = -1$, 0, or 1. [1]

(iii) State with a reason, the number of roots of the equation $w^3 = -8\mathrm{i}$. [1]

(iv) Find the roots of the equation $w^3 = -8\mathrm{i}$ in the form $re^{\mathrm{i}\theta}$ where $-\pi < \theta \le \pi$ and represent them on an Argand diagram. [4]

(v) The locus of the set of points on an Argand diagram represented by the complex number z, given by the equation $|z-a| = r$, describes a circle centered at the complex number a with radius r.
State the values of a and r such that the values of w obtained in part (iv) satisfies the equation $|z-a| = r$. [1]

5 (a) Show algebraically that $\int \ln(a-x)\,\mathrm{d}x$ where a is a real constant, can be expressed as $(x-a)\ln(a-x)-x+c$ for arbitrary constant c. Hence, find $\int (\ln(a-x))^2\,\mathrm{d}x$. [6]

(b) A company manufacturing scientific equipment wants to design a new type of container. A consultant provides them with a design generated by the curve C, given by the equation

$$y = 6 - \mathrm{e}^x.$$

The container is formed by rotating the region bounded by C, the y-axis and the lines $y=0$ and $y=5$, about the y-axis by 2π radians.

(i) Find the exact volume of the container. [4]

(ii) Find the exact vertical cross-sectional area of the container. [2]

Section B: Statistics [60 marks]

6 A digital lock accepts integer inputs from 0 to 9. A 5-digit passcode is to be entered to unlock. The lock accepts unlimited repeats of the digits used in the passcode.

(i) Find the number of passcodes which have no repeated digit. [1]

(ii) Find the number of passcodes with exactly 2 distinct digits that are repeated. An example of such a passcode is "00223". [3]

(iii) Find the probability that a randomly chosen passcode contains repeated digit(s). [2]

7 A bag contains 3 red balls, 4 blue ball and 5 green balls. The balls are identical except for their colours. 5 balls are drawn at random from the bag without replacement and the number of red balls drawn is denoted by X.

(i) Show that $\mathrm{P}(X=2)=\dfrac{7}{22}$.

Determine the probability distribution of X. [4]

(ii) Find the expectation and variance of X. [2]

In a round of a game, the player makes 5 draws from the bag. To start the game, the player has to pay \$3. He gets \$2 for each red ball that he draws.

(iii) Find the value of $\mathrm{E}(2X-3)$ and interpret the significance of this value from the player's perspective. [2]

8 A manufacturer produces 3 types of spray bottles: Type A, Type B and Type C. 65% of the sprayers manufactured are Type A and 20% are Type B. 3% of Type A sprayers, 4% of Type B sprayers and 5% of Type C sprayers have manufacturing defects.

(i) A sprayer is chosen at random. Construct a probability tree to show the above information. [2]

(ii) Find the probability that out of 2 randomly selected sprayers, exactly one of them has manufacturing defects. [3]

(iii) Three sprayers are randomly chosen. Find the probability that there are exactly 2 Type C sprayers given that exactly one of the three sprayers has manufacturing defect. [4]

9 During each round of practice, John, does 10 multiple-choice questions and his score X, is the number of questions he answered correctly. On average, he has an 85% chance of answering each multiple-choice question correctly.

(i) State in the context of the question, one assumption needed to model X by a binomial distribution. [1]

On a particular day, John does 4 rounds of practice.

(ii) Find the probability that the total score for John in the 4 rounds of practice is more than 36. [2]

(iii) Find the probability that John obtains a score of at most 9 in each of his first 2 rounds of practice. [3]

To qualify for an award, John will need a mean score of at least 8.8 for all his 50 rounds of practice.

(iv) Find the probability that John will qualify for the award. [3]

10 In a certain district, the English and Maths marks for Primary 6 students in the national primary school leaving examination follow normal distributions with means and variances shown below:

Subject	Mean	Variance
English	76	25
Maths	74	σ^2

The performance in English is independent of that in Maths for Primary 6 students in the district.

(i) Past data revealed that 1 in every 30 students in the district scored at least 85 marks in Maths in the examination. Find the value of σ correct to the nearest whole number. [2]

For the rest of the question, take σ to be the nearest whole number obtained in part (i).

(ii) To qualify for the top secondary school in the district, a Primary 6 student will need to have the sum of his Maths mark and double of his English mark to be at least 250. Given that a particular student qualifies for the top secondary school, find the probability that he scored more than 85 marks in each of the English and Maths examination. [4]

(iii) The Education Board of the district decides to study if there is any significant difference in the Primary 6 students' performance for the 2 subjects in the examination. A sample of 5 students is chosen from School A in the district and the sum X of their English marks is obtained. Another sample of 5 students is chosen from School B in the same district and the sum Y of their Maths marks is obtained. Find the probability that the difference of X and Y is more than 30 marks. [4]

(iv) A student scoring more than 80 marks in English qualifies for an enrichment course. A sample of n students are chosen. Find the least value of n such that the probability of more than 4 students qualifying for the enrichment course is more than 0.15. [4]

11 In a factory, the time taken in minutes by each worker to assemble the components of an electrical device with adherence to strict production standard is known to have a mean of 15 minutes. The factory management team implements a new assembly procedure.
After the new assembly procedure has been implemented for half a year, the production manager chooses 40 workers randomly and records the time t minutes each worker takes to complete the assembly process. The results collected are summarized as follow

$$\sum(t-15)=-8, \quad \sum(t-15)^2=17.$$

(i) Briefly describe how a random sample of 40 workers could be obtained. [1]

(ii) Calculate the unbiased estimates of the population mean and variance of the time to assemble the components of the electrical device, in the new procedure. [2]

(iii) Test, at the 5% level of significance, whether the mean time in minutes for the worker to assemble the components of the electrical device has changed after the implementation of the new assembly procedure. You should state your hypotheses clearly and define any symbols used. [4]

(iv) Explain the meaning of 'at the 5% level of significance' in the context of the question. [1]

(v) Explain if it is necessary for the production manager to assume that the time a worker takes to complete the assembly process follows a normal distribution. [1]

The production manager believes that the mean time should have been reduced with the implementation of the new assembly procedure.

(vi) Deduce if the collected data support his belief at the 2% level of significance. [2]

The production manager then proceed to further collect a set of assembly times in minutes from another 10 different randomly chosen workers as given below.

14.8, 15.1, 14.9, 14.8, 15, 15.1, 14.8, 14.7, 14.9, 14.9

The above data from the 10 different workers are summarized as follow

$$\sum(t-15)=-1, \quad \sum(t-15)^2=0.26.$$

(vii) Combining the assembly times of the first 40 workers and these additional 10 workers into a single sample, a test at $100\alpha\%$ level of significance supports the production manager's belief that the new assembly procedure has helped to reduce the mean assembly time. Find the range of values of α. [3]

END OF PAPER

RIVER VALLEY HIGH SCHOOL
2021 JC2 Preliminary Examination
Higher 2

NAME	

CLASS							INDEX NUMBER	

MATHEMATICS

9758/01

Paper 1

16 Sep 2021

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total	

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

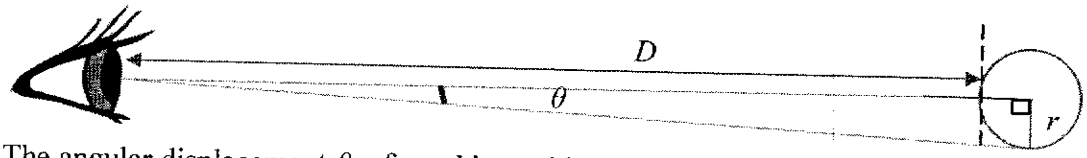
Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Calculator Model:

This document consists of ___ printed pages and ___ blank pages.

1 It is given that $y = \mathrm{f}(x)$ is a function such that $\mathrm{f}(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$. Find the first three non-zero terms in the series expansion of $\frac{\mathrm{f}(x)}{\sqrt{9-x}}$. [4]

2 The *angular displacement* of an object is how wide that object appears to be from an observer's point of view.



The angular displacement θ, of an object with radius r km that is D km away is shown in the diagram above.

Oumuamua is the first interstellar object observed in our solar system. Discovered in 2017, it had an angular displacement of 2.85×10^{-10} radians. Powerful Earth-based lasers measured its distance as 3.59×10^{9} km from Earth.

(i) Show that the radius of *Oumuamua* is approximately 1.023 km, rounded to 3 decimal places. [2]

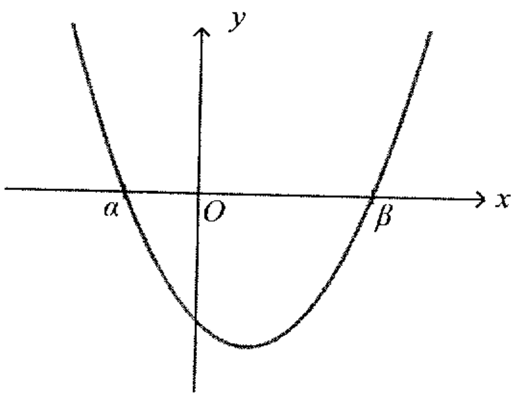
(ii) The rate of change of angular displacement is found to be -1.095×10^{-14} radians per hour at this instant. Assuming θ is sufficiently small for θ^3 and higher powers to be ignored, find the rate at which *Oumuamua* is moving away from Earth. [4]

3 It is given that $1 - \mathrm{i}$ is a root of the equation $2z^4 + pz^3 + 8z^2 + qz + 4 = 0$, where p and q are real constants.

(i) Write down $(1-\mathrm{i})^2$, $(1-\mathrm{i})^3$ and $(1-\mathrm{i})^4$ in cartesian form. Hence find the values of p and q. [3]

(ii) Without the use of calculator, find the other roots of the equation in exact form. [4]

4



The diagram shows the graph of $y = x^2 - 2x - 5$. The two roots of the equation $x^2 - 2x - 5 = 0$ are denoted by α and β, where $\alpha < \beta$.

(i) Find the exact values of α and β. [2]

A sequence of positive numbers $x_1, x_2, x_3, \ldots$ is such that

$$x_{n+1} = (2x_n + 5)^{\frac{1}{2}} \text{ for } n = 1, 2, 3, \ldots.$$

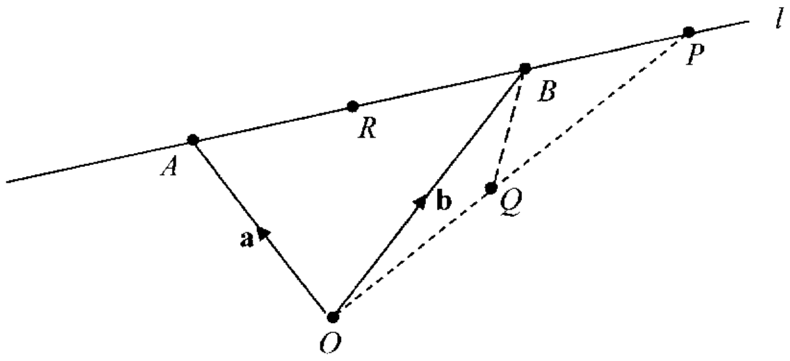
As $n \to \infty$, $x_n \to l$.

(ii) Explain why l satisfies the equation $l^2 - 2l - 5 = 0$. [1]

(iii) Show that $l = \beta$. [2]

(iv) By considering the graph of $y = x^2 - 2x - 5$ and the sign of y, show that when $0 < x_n < \beta$, then $x_n < x_{n+1}$. [2]

5 The diagram below shows a straight line l passing through the points A and B. With reference to the origin O, the position vectors of A and B are **a** and **b** respectively. It is further given that **a** is a unit vector, $|\mathbf{b}| = 2$ and $\angle AOB = 60°$.



(i) State the values of $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|$. [2]

(ii) The point P lies on the line l and is such that $AB : BP = 2 : 1$. The point Q on the line OP is such that $\overrightarrow{OQ} = \lambda\overrightarrow{OP}$ where $0 < \lambda < 1$. Determine the value of λ such that the area of triangle OBQ is $\frac{1}{2\sqrt{3}}$ of the area of triangle OAB. [3]

(iii) It is further given that the point R on the line l is such that $\angle AOR = \angle ROB$. Show that R has position vector $\mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$ for some $\mu \in \mathbb{R}$ and hence find this value of μ. [3]

6 Aeroplanes need to maintain a certain speed in order to generate enough lift to stay aloft. However, moving quickly is inefficient at low altitudes, where there is greater air resistance. There is less air resistance at higher altitudes, but planes need to move faster in order to stay aloft as there is less air.

The following formula relates the speed of a particular aeroplane necessary to stay aloft to the altitude at which it is travelling:

$$v = 0.8h^3 - 18h^2 + 170h,$$

where v is the minimum speed in kilometres per hour, to stay aloft at an altitude of h thousand kilometres.

It is also given that

$$E = \frac{1000(700)^2}{700^2 + (v - 700)^2},$$

where E is the fuel efficiency of a suitable unit, at which the aeroplane is travelling.

(i) Show that $\frac{dv}{dh} > 0$ for all h. [2]

(ii) By differentiation, find the height at which the aeroplane should travel in order to achieve the greatest fuel efficiency. [5]

7 An insurance company, Super Western, launched a new investment plan. The investment plan offers a fixed interest rate of 5% of the amount available in the plan at the start of that year. The interest is added to the plan at the end of each year. Tom and Mary decide to invest in this plan.

(a) Mary decides to place \$$y$ at the start of the first year and then a further \$$y$ again at the start of each subsequent year. She chooses to leave the money in the investment plan and let the interest accumulate.

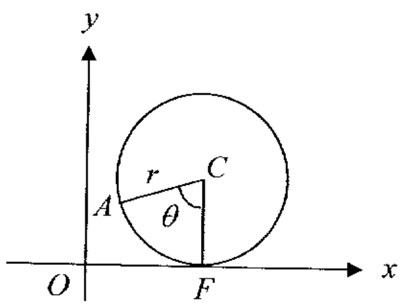
(i) How much money will there be in the investment plan at the end of 1 year? [1]

(ii) Suppose that the interest of the final year has been added into the plan, show that at the end of n years, Mary will have a total amount of $\$21(1.05^n - 1)y$ in her plan. [3]

(iii) Calculate the number of complete years it takes Mary to have at least $\$15y$ in her plan. [2]

(b) Tom decides to utilize the investment plan differently. He plans to withdraw the interest immediately when the interest is being added into his plan. Suppose that Tom invests \$$3x$ at the start of first year and \$$2x$ at start of each subsequent year, what is the total amount of interest he has withdrawn at the end of n years? [3]

8 A wheel with centre C is pushed along a flat surface in a straight line. The point A on the wheel, is initially in contact with the ground at O. After the wheel has rotated through an angle of θ radians, the point of contact with the ground is F and the length of the arc AF is equal to OF.



The wheel has a fixed radius of r.

(a) Show that the coordinates of A after the wheel has rotated through an angle of θ radians is

$$\left(r\theta - r\sin\theta,\ r - r\cos\theta\right).$$

Hence or otherwise, find the cartesian equation of the locus of A as the wheel is pushed along the surface, for $0 \le \theta \le 2\pi$. Express your answer as x in terms of y and r. [4]

(b) The parametric equation of the curve C is given by

$$x = rt - r\sin t$$

$$y = r - r\cos t,$$

for $0 \le t \le 2\pi$, where r is a constant.

(i) Show that $\int y \, dx = r^2 \int (1-\cos t)^2 \, dt$. [2]

(ii) Hence or otherwise, find the exact area of the region bounded by C and the x-axis. [3]

9 The curve C has equation $y = \dfrac{x^2 + x - 2}{x+k}$, where $x \in \mathbb{R}$, $x \ne -k$ for some $k \in \mathbb{R}$.

(i) Find the range of values of k such that C has 2 stationary points. [3]

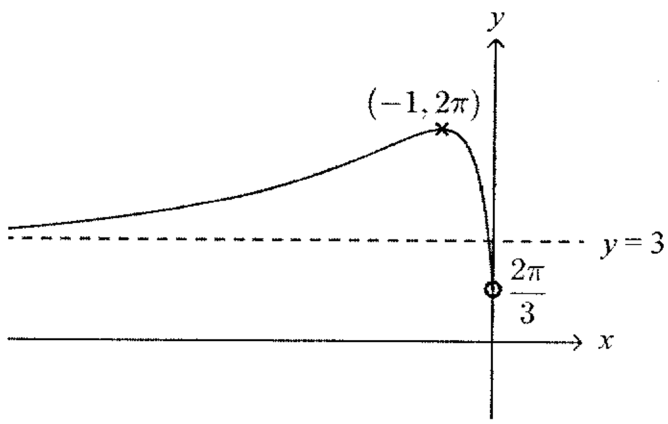
(ii) Sketch C for $k = 3$, stating the equation of any asymptotes, the coordinates of stationary points and points where the curve crosses the axes. [3]

(iii) By adding a suitable graph to the graph in part (ii), solve the inequality $x^4 + x^3 - 2x^2 - x - 3 \le 0$ for values of $x > -3$. [3]

10 The function f is defined by $\mathrm{f} : x \mapsto \sin x + x$, for $x \in \mathbb{R}, x > 0$.

(i) Show that $\mathrm{f}'(x) \ge 0$. Hence, or otherwise, show that f^{-1} exists. [3]

(ii) Find the values of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]



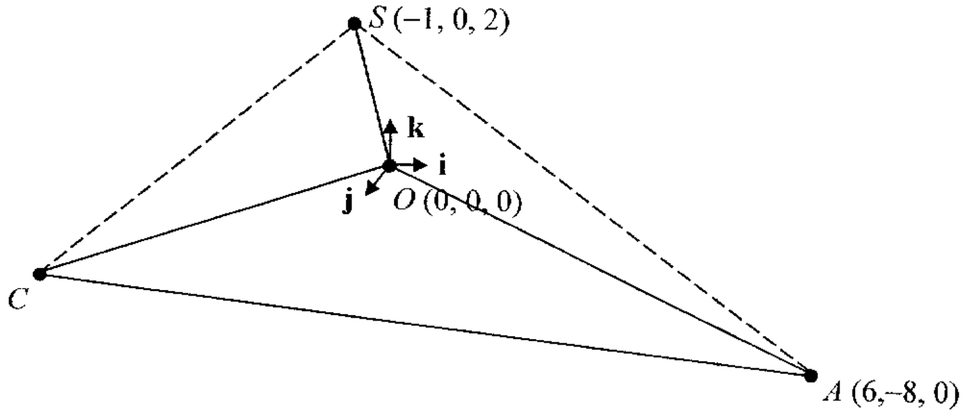
The diagram above shows the graph of $y = \mathrm{g}(x)$.

(iii) Show that the composite function fg exists. Hence, determine the range of fg. [4]

(iv) The function h is defined by $\mathrm{h}(x) = \mathrm{g}(x)$ for $x \le -1$. State the value of $\mathrm{h}^{-1}\mathrm{f}(2\pi)$. [1]

11 In an indoor playground, a virtual reality enclosure is being set up. The base of the enclosure takes on the shape of a triangle OCA as shown in the diagram. The point O represents the origin and the point A has coordinates (6,–8, 0). The highest point of the enclosure is at the point S with coordinates (–1, 0, 2). The walls of the enclosure are represented by triangles SCA, SCO and SAO respectively.

The line that passes through C and A has equation $\frac{6-x}{2}=y+8,\ z=0$. The base of the enclosure is represented by the plane with equation $z=0$.



(i) Find the cartesian equation of the plane representing the wall SCA. Hence, find the acute angle between the base of the enclosure and the wall SCA. [4]

(ii) A laser projector is to be set up at a point F along the line segment CA such that it is closest to the point S. Find the coordinates of F. [3]

(iii) The projector at F emits a laser beam that makes a point on wall SAO to create the illusion of a shooting star. The projected point moves in a straight line on the wall SAO from S to the point N which is closest to F. Find the length of FN. [3]

(iv) Calculate the angle that the laser beam emitted from the projector sweeps through as the point moves from S to N. [2]

12 (i) The drag force experienced by an object moving through a medium can be modelled by the differential equation

$$\frac{dv}{dt}=-\mu v-\lambda v^3,$$

where v is the velocity of the object in metres per second, t is time in seconds, and μ and λ are real positive constants.

Using the substitution $z=\frac{1}{v^2}$, show that $\frac{dz}{dt}=2\mu z+2\lambda$.

Hence, find the general solution for $v>0$, in terms of t. [8]

(ii) State the long term behaviour of v. [1]

For the rest of the question, take $\mu=0.1$ and $\lambda=0.08$.

(iii) Find the particular solution of v for which the object is travelling at 1 ms^{-1} after 3 s. [3]

END OF PAPER

RVHS 2021 H2 MA Prelim Paper 2

Section A: Pure Mathematics [40 marks]

1 The function f is defined by $\mathrm{f}(z) = az^3 + bz^2 + cz + d$ where a, b, c and d are real numbers. Given that $1-\sqrt{3}\,\mathrm{i}$ and $\frac{3}{2}$ are roots of $\mathrm{f}(z) = 0$, find b, c and d in terms of a. [4]

2 (i) By using the formulae for $\sin(A \pm B)$, show that

$$\sin\left(p+\frac{1}{2}\right)\theta - \sin\left(p-\frac{1}{2}\right)\theta \equiv 2\cos p\theta \sin\frac{1}{2}\theta .$$ [1]

(ii) Hence show that $\sum_{p=1}^{n} \cos p\theta = \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\left[\sin\left(n+\frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta\right]$. [2]

(iii) Using the result in part (ii), find the exact value for $\sum_{p=5}^{32} \cos(p-2)\theta$ when $\theta = \frac{\pi}{2}$. [3]

3 A curve C has equation $y^2 + xy = -1$, where $y > -1$.

(a) Describe a sequence of transformations that will transform the graph of C onto the graph of $4y^2 - 2(x-1)y = -1$. [3]

(b) (i) Show that the gradient function of C can be expressed as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{2y+x}$. [2]

(ii) For a non-zero constant a, find the equation of the tangent to the point where $y = a$ in terms of a. [3]

(iii) State the behaviour of the tangent as $a \to 1$. [1]

4 (i) Verify that $8\mathrm{e}^{\mathrm{i}\left(2k\pi - \frac{\pi}{2}\right)} = -8\mathrm{i}$, where k is an integer. [2]

(ii) Given that $\theta = \frac{2k\pi}{3} - \frac{\pi}{6}$ where k is an integer and $-\pi < \theta \le \pi$, show that $k = -1$, 0, or 1. [1]

(iii) State with a reason, the number of roots of the equation $w^3 = -8\mathrm{i}$. [1]

(iv) Find the roots of the equation $w^3 = -8\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $-\pi < \theta \le \pi$ and represent them on an Argand diagram. [4]

(v) The locus of the set of points on an Argand diagram represented by the complex number z, given by the equation $|z-a| = r$, describes a circle centered at the complex number a with radius r.
State the values of a and r such that the values of w obtained in part (iv) satisfies the equation $|z-a| = r$. [1]

5 (a) Show algebraically that $\int \ln(a-x)\,\mathrm{d}x$ where a is a real constant, can be expressed as $(x-a)\ln(a-x)-x+c$ for arbitrary constant c. Hence, find $\int (\ln(a-x))^2\,\mathrm{d}x$. [6]

(b) A company manufacturing scientific equipment wants to design a new type of container. A consultant provides them with a design generated by the curve C, given by the equation

$$y=6-\mathrm{e}^x.$$

The container is formed by rotating the region bounded by C, the y-axis and the lines $y=0$ and $y=5$, about the y-axis by 2π radians.

(i) Find the exact volume of the container. [4]

(ii) Find the exact vertical cross-sectional area of the container. [2]

Section B: Statistics [60 marks]

6 A digital lock accepts integer inputs from 0 to 9. A 5-digit passcode is to be entered to unlock. The lock accepts unlimited repeats of the digits used in the passcode.

(i) Find the number of passcodes which have no repeated digit. [1]

(ii) Find the number of passcodes with exactly 2 distinct digits that are repeated. An example of such a passcode is "00223". [3]

(iii) Find the probability that a randomly chosen passcode contains repeated digit(s). [2]

7 A bag contains 3 red balls, 4 blue ball and 5 green balls. The balls are identical except for their colours. 5 balls are drawn at random from the bag without replacement and the number of red balls drawn is denoted by X.

(i) Show that $\mathrm{P}(X=2)=\frac{7}{22}$.

Determine the probability distribution of X. [4]

(ii) Find the expectation and variance of X. [2]

In a round of a game, the player makes 5 draws from the bag. To start the game, the player has to pay \$3. He gets \$2 for each red ball that he draws.

(iii) Find the value of $\mathrm{E}(2X-3)$ and interpret the significance of this value from the player's perspective. [2]

8 A manufacturer produces 3 types of spray bottles: Type A, Type B and Type C. 65% of the sprayers manufactured are Type A and 20% are Type B. 3% of Type A sprayers, 4% of Type B sprayers and 5% of Type C sprayers have manufacturing defects.

(i) A sprayer is chosen at random. Construct a probability tree to show the above information. [2]

(ii) Find the probability that out of 2 randomly selected sprayers, exactly one of them has manufacturing defects. [3]

(iii) Three sprayers are randomly chosen. Find the probability that there are exactly 2 Type C sprayers given that exactly one of the three sprayers has manufacturing defect. [4]

9 During each round of practice, John, does 10 multiple-choice questions and his score X, is the number of questions he answered correctly. On average, he has an 85% chance of answering each multiple-choice question correctly.

(i) State in the context of the question, one assumption needed to model X by a binomial distribution. [1]

On a particular day, John does 4 rounds of practice.

(ii) Find the probability that the total score for John in the 4 rounds of practice is more than 36. [2]

(iii) Find the probability that John obtains a score of at most 9 in each of his first 2 rounds of practice. [3]

To qualify for an award, John will need a mean score of at least 8.8 for all his 50 rounds of practice.

(iv) Find the probability that John will qualify for the award. [3]

10 In a certain district, the English and Maths marks for Primary 6 students in the national primary school leaving examination follow normal distributions with means and variances shown below:

Subject	Mean	Variance
English	76	25
Maths	74	σ^2

The performance in English is independent of that in Maths for Primary 6 students in the district.

(i) Past data revealed that 1 in every 30 students in the district scored at least 85 marks in Maths in the examination. Find the value of σ correct to the nearest whole number. [2]

For the rest of the question, take σ to be the nearest whole number obtained in part (i).

(ii) To qualify for the top secondary school in the district, a Primary 6 student will need to have the sum of his Maths mark and double of his English mark to be at least 250. Given that a particular student qualifies for the top secondary school, find the probability that he scored more than 85 marks in each of the English and Maths examination. [4]

(iii) The Education Board of the district decides to study if there is any significant difference in the Primary 6 students' performance for the 2 subjects in the examination. A sample of 5 students is chosen from School A in the district and the sum X of their English marks is obtained. Another sample of 5 students is chosen from School B in the same district and the sum Y of their Maths marks is obtained. Find the probability that the difference of X and Y is more than 30 marks. [4]

(iv) A student scoring more than 80 marks in English qualifies for an enrichment course. A sample of n students are chosen. Find the least value of n such that the probability of more than 4 students qualifying for the enrichment course is more than 0.15. [4]

11 In a factory, the time taken in minutes by each worker to assemble the components of an electrical device with adherence to strict production standard is known to have a mean of 15 minutes. The factory management team implements a new assembly procedure.
After the new assembly procedure has been implemented for half a year, the production manager chooses 40 workers randomly and records the time t minutes each worker takes to complete the assembly process. The results collected are summarized as follow

$$\sum(t-15)=-8, \quad \sum(t-15)^2=17.$$

(i) Briefly describe how a random sample of 40 workers could be obtained. [1]

(ii) Calculate the unbiased estimates of the population mean and variance of the time to assemble the components of the electrical device, in the new procedure. [2]

(iii) Test, at the 5% level of significance, whether the mean time in minutes for the worker to assemble the components of the electrical device has changed after the implementation of the new assembly procedure. You should state your hypotheses clearly and define any symbols used. [4]

(iv) Explain the meaning of 'at the 5% level of significance' in the context of the question. [1]

(v) Explain if it is necessary for the production manager to assume that the time a worker takes to complete the assembly process follows a normal distribution. [1]

The production manager believes that the mean time should have been reduced with the implementation of the new assembly procedure.

(vi) Deduce if the collected data support his belief at the 2% level of significance. [2]

The production manager then proceed to further collect a set of assembly times in minutes from another 10 different randomly chosen workers as given below.

14.8, 15.1, 14.9, 14.8, 15, 15.1, 14.8, 14.7, 14.9, 14.9

The above data from the 10 different workers are summarized as follow

$$\sum(t-15)=-1, \quad \sum(t-15)^2=0.26.$$

(vii) Combining the assembly times of the first 40 workers and these additional 10 workers into a single sample, a test at $100\alpha\%$ level of significance supports the production manager's belief that the new assembly procedure has helped to reduce the mean assembly time. Find the range of values of α. [3]

END OF PAPER

2021 JC2 H2 Maths Prelim Paper 1

1	Solution [4] Mcluarin	
	$\frac{dy}{dx}=1+y^2$ $\frac{d^2y}{dx^2}=2y\frac{dy}{dx}$ When $x=0$, $y=1$, $\frac{dy}{dx}=2$, $\frac{d^2y}{dx^2}=4$ $f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+...$ $\approx 1+2x+2x^2$ $\frac{f(x)}{\sqrt{9-x}}$ $\approx \frac{1+2x+2x^2}{\sqrt{9-x}}$ $=(1+2x+2x^2)(9-x)^{-\frac{1}{2}}$ $=(1+2x+2x^2)(9)^{-\frac{1}{2}}\left(1-\frac{x}{9}\right)^{-\frac{1}{2}}$ $\approx \frac{1}{3}(1+2x+2x^2)\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{9}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{9}\right)^2\right]$ $=\frac{1}{3}(1+2x+2x^2)\left[1+\frac{1}{18}x+\frac{1}{216}x^2\right]$ $=\frac{1}{3}\left[1+\frac{1}{18}x+\frac{1}{216}x^2+2x+\frac{2}{18}x^2+2x^2\right]$ $=\frac{1}{3}\left[1+\frac{37}{18}x+\frac{457}{216}x^2\right]$	Some students are careless in their algebraic manipulations and simplification of the terms. Some of the common errors are: $\frac{d^2y}{dx^2}=1+2y\frac{dy}{dx}$ $\frac{f(x)}{\sqrt{9-x}}=f(x)(9-x)^{\frac{1}{2}}$ $\frac{f(x)}{\sqrt{9-x}}=f(x)(3)\left(1-\frac{x}{9}\right)^{-\frac{1}{2}}$

2	Solution [6] Rate of Change	
(i)	$\tan\theta = \frac{r}{D+r}$ $2.85\times10^{-10} = \frac{r}{3.59\times10^{9}+r}$ $1.02315+\left(2.85\times10^{-10}\right)r = r$ $r = \frac{1.02315}{1-2.85\times10^{-10}} = 1.02315$ $r \approx 1.023$	Not well attempted as students did not noticed that the adjacent length is $D+r$. While final answer can be the same, due to the wrong understanding of the question, the marks are not awarded.
(ii)	$\frac{d\theta}{dt} = -1.095\times10^{-14}$ $\frac{dD}{dt} = \frac{dD}{d\theta}\frac{d\theta}{dt}$ $\tan\theta \approx \theta$ $\theta \approx \frac{r}{r+D}$ $D \approx \frac{r}{\theta} - r$ $\frac{dD}{d\theta} = \frac{-r}{\theta^2}$ $\frac{dD}{dt} = \frac{dD}{d\theta}\frac{d\theta}{dt}$ $= \left(\frac{1.023}{\left(2.85\times10^{-10}\right)^2}\right)\left(-1.095\times10^{-14}\right)$ $= 137911.3573$ $\approx 138\ 000$ km/h	Some of the students do not understand the question and do not know where to start. They also failed to see the "small angle" to do the approximation for $\tan\theta$. Like in part (i), they failed to see the adjacent length is $D+r$

3	Solution [7] Complex Number	
(i)	$z^2=(1-\mathrm{i})^2=1-2\mathrm{i}+\mathrm{i}^2=-2\mathrm{i}$ $z^3=(1-\mathrm{i})(-2\mathrm{i})=-2-2\mathrm{i}$ $z^4=(1-\mathrm{i})(-2-2\mathrm{i})=-2-2\mathrm{i}+2\mathrm{i}-2=-4$ Since one of the roots of the equation is $1-\mathrm{i}$ $2(1-\mathrm{i})^4+p(1-\mathrm{i})^3+8(1-\mathrm{i})^2+q(1-\mathrm{i})+4=0$ $2(-4)+p(-2-2\mathrm{i})+8(-2\mathrm{i})+q(1-\mathrm{i})+4=0$ $-8-2p-2p\mathrm{i}-16\mathrm{i}+q-q\mathrm{i}+4=0$ $(-4-2p+q)+(-2p-16-q)\mathrm{i}=0$ ....(*) Re: $-2p+q=4$ Im: $-2p-q=16$ $p=-5$ $q=-6$	Some of the students do not know the different forms of the complex number. Students can simply use their calculators to evaluate these answers. Some of the students are careless in their working and failed to get the correct answer despite getting the correct equations.
(ii)	As the coefficients are all real, complex roots occur in conjugate pairs, $\therefore$ Another root is $z=1+\mathrm{i}$. $2z^4-5z^3+8z^2-6z+4=[z-(1-\mathrm{i})][z-(1+\mathrm{i})](2z^2+az+b)$ $[z-(1-\mathrm{i})][z-(1+\mathrm{i})](2z^2+az+b)$ $=[(z-1)+\mathrm{i}][(z-1)-\mathrm{i}](2z^2+az+b)$ $=(z^2-2z+2)(2z^2+az+b)$ By comparing coefficients : $2z^4-5z^3+8z^2-6z+4=(z^2-2z+2)(2z^2+az+b)$ $2b=4\Rightarrow b=2$ $2a-2b=-6\Rightarrow 2a-2(2)=-6$ $\therefore a=-1$ Solving for other roots: $2z^2-z+2=0$ $\Rightarrow z=\dfrac{1\pm\sqrt{1-16}}{2(2)}=\dfrac{1\pm\sqrt{-15}}{4}$	Quite a lot of the students simply state that $z=1+\mathrm{i}$ is the other root without stating the reason or giving the wrong reasons. Some of the students failed to simplify correctly the equation due to their handwriting. Some of the students did not use the quadratic equation correctly with the substitution of the various terms.

	The other 2 roots are $\frac{1+\sqrt{15}i}{4}$ or $\frac{1-\sqrt{15}i}{4}$	

4	Solution [7] Sequence	
(i)	$x^2-2x-5=0$ $x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-5)}}{2(1)}=1\pm\sqrt{6}$ $\therefore \alpha=1-\sqrt{6}, \beta=1+\sqrt{6}$	Generally quite well attempted except some careless algebraic manipulations of the terms. However some students did not indicate the values of α and β.
(ii)	$x_{n+1}=(2x_n+5)^{\frac{1}{2}}$ $\lim_{n\to\infty} x_{n+1}^2=\lim_{n\to\infty}(2x_n+5)$. $\therefore l^2=2l+5$ $\Rightarrow l^2-2l-5=0$	The work from the cohort is mixed with some students not able to understand the use of $x_n \to l$ and $x_{n+1} \to l$ as $n \to \infty$ and hence was unable to continue.
(iii)	From (ii), $l^2-2l-5=0$ From (i), $l=\alpha$ or $l=\beta$ Since $\alpha<0$ but $x_n>0$, $l\neq\alpha$. Therefore, $l=\beta$ (shown)	The understanding of the problem is mixed.
(iv)	Since $0<x_n<\beta$, $x_n^2-2x_n-5<0$ from graph $x_n^2<2x_n+5$ $x_n<(2x_n+5)^{\frac{1}{2}}=x_{n+1}$ since $x_n>0$ $x_n<x_{n+1}$ (shown)	This is not well attempted as students did not read the question carefully and did not use the sign of y within the stated region.

5	Solution [8] Abstract Vectors	
(i)	Based on the info provided, we have $\|\mathbf{a}\|=1,\ \|\mathbf{b}\|=2$, $\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}=\|\mathbf{a}\|\|\mathbf{b}\|\cos 60^\circ=1\times2\times\frac{1}{2}=1$ $\|\mathbf{a}\times\mathbf{b}\|=\left\|\|\mathbf{a}\|\|\mathbf{b}\|\sin 60^\circ\hat{\mathbf{n}}\right\|=1\times2\times\frac{\sqrt{3}}{2}\times1=\sqrt{3}$	Generally well attempted except for some careless evaluation of the terms.
(ii)	Since $AB:BP=2:1$, by Ratio Theorem, $\overrightarrow{OB}=\frac{1}{3}\overrightarrow{OA}+\frac{2}{3}\overrightarrow{OP}$ $\Rightarrow 3\overrightarrow{OB}=\overrightarrow{OA}+2\overrightarrow{OP}$ $\Rightarrow \overrightarrow{OP}=\frac{3}{2}\mathbf{b}-\frac{1}{2}\mathbf{a}$ Area of ΔOBQ is $\frac{1}{2\sqrt{3}}$ of the area of ΔOAB. $\frac{1}{2}\left\|\mathbf{b}\times\lambda\left(\frac{3}{2}\mathbf{b}-\frac{1}{2}\mathbf{a}\right)\right\|=\frac{1}{2\sqrt{3}}\left[\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|\right]$ $\frac{1}{2}\left\|\mathbf{b}\times\frac{3\lambda}{2}\mathbf{b}-\mathbf{b}\times\frac{\lambda}{2}\mathbf{a}\right\|=\frac{1}{4\sqrt{3}}\|\mathbf{a}\times\mathbf{b}\|$ $\frac{1}{2}\left\|\mathbf{b}\times\frac{\lambda}{2}\mathbf{a}\right\|=\frac{1}{4\sqrt{3}}\|\mathbf{a}\times\mathbf{b}\|$ $\frac{\lambda}{4}\|\mathbf{a}\times\mathbf{b}\|=\frac{1}{4\sqrt{3}}\|\mathbf{a}\times\mathbf{b}\|$ $\lambda=\frac{1}{\sqrt{3}}$	Generally not well attempted by the students as they failed to find $\overrightarrow{OP}$ which is helpful to obtain the area of ΔOBQ
(iii)	We first note that since A, R and B are collinear, for some $\mu\in\mathbb{R}$, we have $\overrightarrow{AR}=\mu\overrightarrow{AB}$. Then $\overrightarrow{OR}-\overrightarrow{OA}=\mu\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$ Thus, $\overrightarrow{OR}=\overrightarrow{OA}+\mu\left(\overrightarrow{OB}-\overrightarrow{OA}\right)=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a})$ Since $\angle AOR=\angle ROB$, $\cos\angle AOR=\cos\angle ROB$ $\frac{\overrightarrow{OA}\bullet\overrightarrow{OR}}{\|\overrightarrow{OA}\|\|\overrightarrow{OR}\|}=\frac{\overrightarrow{OR}\bullet\overrightarrow{OB}}{\|\overrightarrow{OR}\|\|\overrightarrow{OB}\|}$ $\frac{\mathbf{a}\bullet(\mathbf{a}+\mu(\mathbf{b}-\mathbf{a}))}{1}=\frac{(\mathbf{a}+\mu(\mathbf{b}-\mathbf{a}))\bullet\mathbf{b}}{2}$	Not well attempted as students see the $\angle AOR=\angle ROB$ as R is the midpoint of A and B instead.

	$2(\mathbf{a}\bullet\mathbf{a}+\mu\mathbf{a}\bullet\mathbf{b}-\mu\mathbf{a}\bullet\mathbf{a})=\mathbf{a}\bullet\mathbf{b}+\mu\mathbf{b}\bullet\mathbf{b}-\mu\mathbf{a}\bullet\mathbf{b}$ $2\left(\lvert\mathbf{a}\rvert^2+\mu\mathbf{a}\bullet\mathbf{b}-\mu\lvert\mathbf{a}\rvert^2\right)=\mathbf{a}\bullet\mathbf{b}+\mu\lvert\mathbf{b}\rvert^2-\mu\mathbf{a}\bullet\mathbf{b}$ $2(1+\mu-\mu)=1+4\mu-\mu$ $\Rightarrow 3\mu=2-1$ $\Rightarrow \mu=\frac{1}{3}$.	

6	Solution [7] Maxima and Minima	
(i)	$v = 0.8h^3 - 18h^2 + 170h$ $\frac{dv}{dh} = 2.4h^2 - 36h + 170$ $= 2.4\left(h^2 - 15h\right) + 170$ $= 2.4\left(\left(h - 7.5\right)^2 - 56.25\right) + 170$ $= 2.4\left(h - 7.5\right)^2 + 35 > 0$ Alternative (discriminant method) $\frac{dv}{dh} = 2.4h^2 - 36h + 170$ $D = b^2 - 4ac$ $= (-36)^2 - 4(2.4)(170)$ $= -336 < 0$ And since $a = 2.4 > 0$, $\frac{dv}{dh} > 0$	For students who used the discriminant method, many did not consider the coefficient of h^2.
(ii)	To find max/min set $\frac{dE}{dh} = 0$, $\frac{dE}{dh} = \frac{dE}{dv}\frac{dv}{dh}$ and since $\frac{dv}{dh} > 0$, solve $\frac{dE}{dv} = 0$ $E = \frac{1000(700)^2}{700^2 + (v - 700)^2}$ $\frac{dE}{dv} = \frac{2000(700)^2(700 - v)}{\left[700^2 + (v - 700)^2\right]^2}$ $0 = 700 - v$ $v = 700$ From GC, when $v = 700$ $\frac{d^2E}{dv^2} = -2000 < 0$ Therefore, E is a maximum. When $v = 700$ $700 = 0.8h^3 - 18h^2 + 170h$ $0 = 0.8h^3 - 18h^2 + 170h - 700$ $h = 10$	Though the differentiation may be tedious, students who persevere tend to get most of the marks. Many did not use chain rule to get relationship between E, v and h. Instead, students obtained a relationship between E and h that was quite messy. Many did not check for maximum. This to be done through first or second derivative test. In either test, the values of the first or second derivative must be written down. Writing merely > 0, < 0, / or \ is not enough. For first derivative test, should check dE/dv at v = 699.5, 700 and 700.5. Final answer is 10 *thousand* km. Many did not

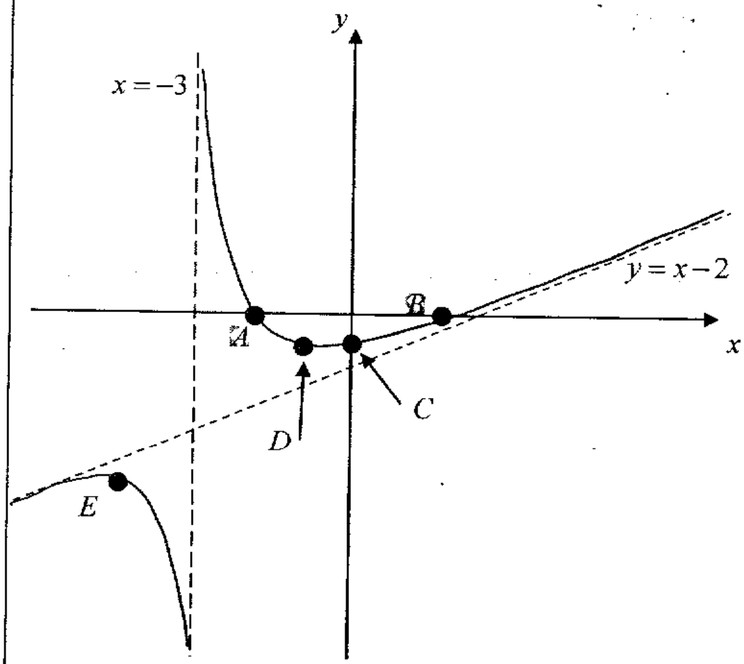
	Therefore height required is 10 000 km.	realise that h is in thousands!

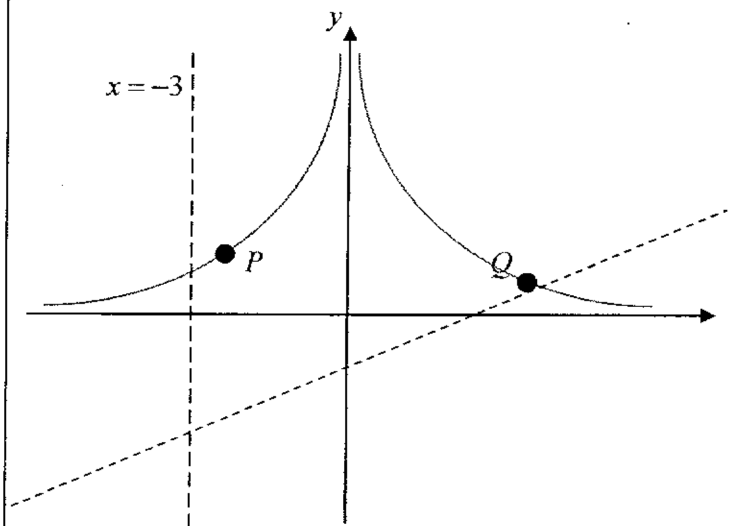
<table>
<tr><td>7</td><td>Solution [9] AP GP</td><td></td></tr>
<tr><td>(a)
(i)</td><td>Amount in account at the end of 1st year:
$= 0.05y + y$
$= 1.05y$</td><td></td></tr>
<tr><td>(a)
(ii)</td><td>
<table>
<tr><td>Year</td><td>Amount received in an account at the end of the year:</td></tr>
<tr><td>1</td><td>$1.05y$</td></tr>
<tr><td>2</td><td>$\left[1.05y + y\right] \times 1.05$
$= 1.05^2 y + 1.05y$</td></tr>
<tr><td>3</td><td>$\left[1.05^2 y + 1.05y + y\right] \times 1.05$
$= 1.05^3 y + 1.05^2 y + 1.05y$</td></tr>
<tr><td>.
.
.</td><td>.
.
.</td></tr>
<tr><td>nth year</td><td>At the end of nth year:
$1.05^n y + 1.05^{n-1} y + \dots + 1.05y$
$= \dfrac{1.05y(1.05^n - 1)}{1.05 - 1}$
$= 21y(1.05^n - 1)$
(Shown)</td></tr>
</table>
</td><td>Generally, students did well in this part, except a few who used y instead if 1.05y as the first term of the GP and led to a wrong answer.</td></tr>
<tr><td>(a)
(iii)</td><td>$21y(1.05^n - 1) \geq 15y$
Since $y > 0$, we can cancel y on both sides
$21(1.05^n - 1) \geq 15$
$1.05^n \geq \dfrac{12}{7}$
$n \geq 11.05$ (to 2dp)
After 12 complete years, Mary would have at least 15y$ in her plan.</td><td>Certainly more appropriate to use "≥" here, instead of "=", since question said "at least $15y".</td></tr>
<tr><td>(b)</td><td>At the end of n years, the interest that Tom received:</td><td>Many students might have misunderstood the question. The question was asking about the total interest withdrawn, not the total amount left. Thus, the factor 0.05 is crucial.</td></tr>
</table>

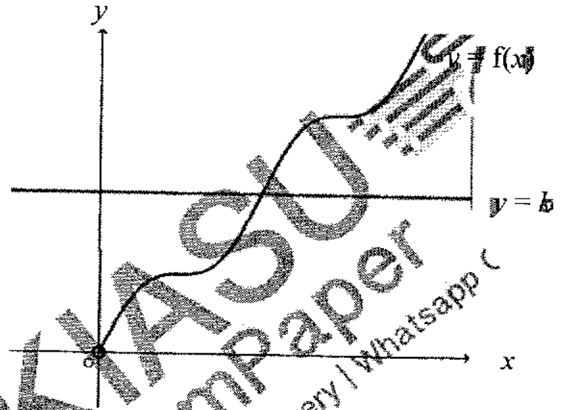
	$0.05(3x)+0.05(5x)+0.05(7x)+...+$ n th term $=0.05x\left[3+5+...+n\text{th term}\right]$ $=0.05x\left[\frac{n}{2}(2(3)+(n-1)(2))\right]$ $=0.05x\left(2n+n^2\right)$	For students who listed some terms and identified the correct general form, some were unable to apply the sum of AP formula correctly. Mistakes included wrong formula, wrong first term or common difference. A good practice is to factorize out the common factors so that you could concentrate on identifying the pattern in the remaining terms. Many also ended the general form with " + ..." which said that the series was infinite. The series was in fact finite!

8	Solution [9] Applications of Differentiation	
(a)	Let B be the foot of perpendicular from A to CF. OF = Arc AF = $r\theta$ By triangle ABC, AB = $r\sin\theta$ and CB = $r\cos\theta$. x-coordinate of A = OF – AB = $r\theta - r\sin\theta$ y-coordinate of A = CF – CB = $r - r\cos\theta$ Thus, $A(r\theta - r\sin\theta, r - r\cos\theta)$ $x = r\theta - r\sin\theta$ $y = r - r\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{r-y}{r}$ $\Rightarrow \quad \sin\theta = \frac{\sqrt{r^2-(r-y)^2}}{r} = \frac{\sqrt{2ry - y^2}}{r}$ $\therefore x = r\cos^{-1}\frac{r-y}{r} - \sqrt{2ry - y^2}$	Many left this part blank. Need to approach this part in the geometric way with use of simple trigo. For Cartesian equation, many tried but left answer in terms of θ.
(b) (i)	$\int y \,\mathrm{d}x = \int y\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$ $= \int (r - r\cos t)(r - r\cos t)\,\mathrm{d}t$ $= r^2 \int (1-\cos t)^2\,\mathrm{d}t$	
(b) (ii)	Area $= r^2 \int_0^{2\pi} (1-\cos t)^2\,\mathrm{d}t$ $= r^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,\mathrm{d}t$ $= r^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right)\mathrm{d}t$ $= r^2 \left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$ $= 3\pi r^2$	Most could integrate $\cos^2 t$ though some gave $\int \cos^2 t\, dt = \frac{\cos^3 t}{3} + c$. Note that $0 \le x \le 2\pi r$ whereas $0 \le t \le 2\pi$.

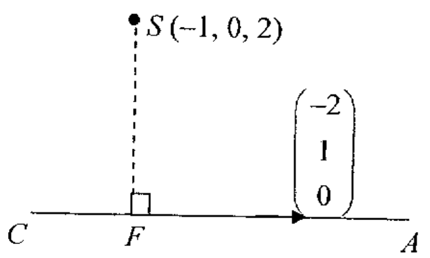
9	Solution [9] Curve Sketching	
(i)	Given $y = \frac{x^2 + x - 2}{x+k}$, we have	Surprisingly quite badly done. Many students were unable to determine the inequality sign of the

	$\frac{dy}{dx}=\frac{(x+k)(2x+1)-(x^2+x-2)(1)}{(x+k)^2}$ $=\frac{x^2+2kx+(k+2)}{(x+k)^2}$ Then for the curve to have 2 stationary points, we need $\frac{dy}{dx}=0$ and hence, $x^2+2kx+(k+2)=0$ to have 2 distinct real roots. Thus, $D=(2k)^2-4(1)(k+2)>0$ $4k^2-4k-8>0$ $k^2-k-2>0$ $(k+1)(k-2)>0$ So the required range of values of k is $k<-1$ or $k>2$.	discriminant for a graph with 2 stationary points.
(ii)	Sketch of $y=\frac{x^2+x-2}{x+3}=x-2+\frac{4}{x+3}$:  The x-intercepts are at points: $A(-2,0)$; $B(1,0)$ The y-intercept is at point $C\left(0,-\frac{2}{3}\right)$ The minimum pt is $D(-1,-1)$; maximum pt is $E(-5,-9)$	Quite badly done as many students have missed out the point C.
(iii)	We first note that for $x>-3$:	Very badly done. Many students were unable to

	$x^4+x^3-2x^2-x-3\le 0$ $\Rightarrow x^4+x^3-2x^2\le x+3$ $\Rightarrow x^2\left(x^2+x-2\right)\le x+3$ $\Rightarrow \frac{x^2+x-2}{x+3}\le\frac{1}{x^2}$ We thus add the additional graph of the curve $C_1: y=\frac{1}{x^2}$ for $x\in \mathrm{R}\setminus\{0\}$  From GC the x coordinates of intersecting points P and Q are -2.07 and 1.54 respectively. Thus, for $\frac{x^2+x-2}{x+3}\le\frac{1}{x^2}$, $-2.07\le x<0$ or $0<x\le 1.54$. Since when $x=0$, $x^4+x^3-2x^2-x-3\le 0$, the solution to the inequality $x^4+x^3-2x^2-x-3\le 0$ is $-2.07\le x\le 1.54$.	manipulate the expression/understand the question to rearrange the expression to obtain a part of the expression similar to previous part. Many only divided the inequality with $x+3$ and plot the wrong graph. There were also some who failed to plot the wrong $y=\frac{1}{x^2}$ without extending the graph beyond the asymptote $x=-3$

10	Solution [10] Functions	
(i)	$\mathrm{f}(x)=\sin x+x$ $\mathrm{f}'(x)=\cos x+1$ Since $-1\le\cos x\le 1$, $0\le\cos x+1\le 2$. Therefore $\mathrm{f}'(x)\ge 0$. (shown) Hence Since $\mathrm{f}'(x)\ge 0$, the function is an increasing function. Therefore f is a one-one function, So f^{-1} exists. <u>Alternative</u>  The line $y=k$, where $k\in\mathbb{R}$ cuts the graph of $y=\mathrm{f}(x)$ at most once. Hence f is one-one and f^{-1} exists.	It is quite surprising that quite a bunch of students were unable to differentiate the function $\cos x+1$. Many were unable to properly explain the reason why $\mathrm{f}'(x)\ge 0$ from $-1\le\cos x\le 1$. Quite a few students wrote the vertical line test instead of stating the correct one.
(ii)	Since the intersections happen on the line $y=x$, we can solve $\mathrm{f}(x)=x$ for the intersections. $\sin x+x=x$ $\sin x=0$ $x=k\pi$, where $k\in\mathbb{Z}, k>0$	Badly done. Many either left it blank or failed to realize the correct conditions required for the values of *k*.
(iii)	$R_{\mathrm{g}}=\left(\frac{2\pi}{3},2\pi\right]\subset(0,\infty)=D_{\mathrm{f}}$ so fg exists.	Averagely done. Still many students were unable to identify the correct range of values for the function *g*. Once range of function *g* is incorrect, many hence were unable to obtain the correct values of the range of

	$D_{fg}=(-\infty,0)\overset{g}{\mapsto}\left(\frac{2\pi}{3},2\pi\right]\overset{f}{\mapsto}\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3},2\pi\right]=R_{fg}$	composite function.
(iv)	$h^{-1}f(2\pi)=h^{-1}(2\pi)=-1$ from graph	Quite well done.

11	Solution [12] 3D Vectors	
(i)	Line CA has Cartesian equation: $\dfrac{6-x}{2}=y+8,\ z=0$, Then, $\dfrac{x-6}{-2}=\dfrac{y-(-8)}{1},\ z=0$ $\Rightarrow$ line CA has vector equation $\mathbf{r}=\begin{pmatrix}6\\-8\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\0\end{pmatrix},\ \lambda\in\mathbf{R}$ Then we have: $\overline{SA}=\overline{OA}-\overline{OS}=\begin{pmatrix}6\\-8\\0\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}7\\-8\\-2\end{pmatrix}$ $\mathbf{n}=\begin{pmatrix}7\\-8\\-2\end{pmatrix}\times\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}2\\4\\-9\end{pmatrix}$ Thus, for vector equation of plane SCA in scalar product form, $\mathbf{r}\bullet\begin{pmatrix}2\\4\\-9\end{pmatrix}=\begin{pmatrix}-1\\0\\2\end{pmatrix}\bullet\begin{pmatrix}2\\4\\-9\end{pmatrix}=-2+0-18=-20$ So, the Cartesian equation of the plane SCA is $2x+4y-9z=-20$ Let θ be the angle between the plane SCA and the ground, $z=0$. $\cos\theta=\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\bullet\begin{pmatrix}2\\4\\-9\end{pmatrix}\right\|}{\sqrt{1}\sqrt{101}}$ $\theta=26.4^{o}$	Most students are able to find the normal of the plane SCA. A number gives the equation of plane SCA in scalar product form instead of Cartesian form. A number of students are not aware that $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the normal of the plane $z=0$.
(ii)	Let F be the point on the line CA that is nearest to pt S  Then the SF is perpendicular to the line CA.	

Since F is a point on the line CA,

$$\overrightarrow{OF}=\begin{pmatrix}6\\-8\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\0\end{pmatrix} \text{ for some } \lambda\in\mathbb{R}$$

We have $\overrightarrow{SF}=\overrightarrow{OF}-\overrightarrow{OS}=\begin{pmatrix}6-2\lambda\\-8+\lambda\\0\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}7-2\lambda\\-8+\lambda\\-2\end{pmatrix}$.

Then since SF is perpendicular to the line CA,

$$\begin{pmatrix}7-2\lambda\\-8+\lambda\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\0\end{pmatrix}=0\Rightarrow -14+4\lambda-8+\lambda=0$$

$$\Rightarrow 5\lambda=22\Rightarrow\lambda=4.4$$

Thus, we have $\overrightarrow{OF}=\begin{pmatrix}6\\-8\\0\end{pmatrix}+4.4\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}-2.8\\-3.6\\0\end{pmatrix}$

and the coordinates of the point F are $(-2.8,-3.6,0)$

Alternatively

$$\overrightarrow{AS}=\begin{pmatrix}-1\\0\\2\end{pmatrix}-\begin{pmatrix}6\\-8\\0\end{pmatrix}=\begin{pmatrix}-7\\8\\2\end{pmatrix}$$

$\overrightarrow{OF}$

$=\overrightarrow{OA}+\overrightarrow{AF}$

$=\overrightarrow{OA}+\left(\overrightarrow{AS}\bullet\hat{\mathbf{m}}\right)\hat{\mathbf{m}}$

$$=\begin{pmatrix}6\\-8\\0\end{pmatrix}+\left[\begin{pmatrix}-7\\8\\2\end{pmatrix}\bullet\frac{1}{\sqrt5}\begin{pmatrix}-2\\1\\0\end{pmatrix}\right]\left[\frac{1}{\sqrt5}\begin{pmatrix}-2\\1\\0\end{pmatrix}\right]$$

$$=\begin{pmatrix}6\\-8\\0\end{pmatrix}+\frac{1}{5}\left[\begin{pmatrix}-7\\8\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\0\end{pmatrix}\right]\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$=\begin{pmatrix}6\\-8\\0\end{pmatrix}+\frac{22}{5}=\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}-2.8\\-3.6\\0\end{pmatrix}$$

Coordinates of the point F are $(-2.8,-3.6,0)$

Some students leave this part blank, not realizing that point F on line CA being closest to point S, means that F is the foot of perpendicular from S on line CA.

Students who are aware that they are tasked to find the foot of perpendicular, generally score well.

For those who attempt using this method, they applied the projection of vector incorrectly as

$\overrightarrow{AF}=\left(\overrightarrow{SA}\bullet\hat{\mathbf{m}}\right)\hat{\mathbf{m}}$

Students should draw clear diagram to avoid above mmistake.

There are a number of arithmetic mistakes.

(iii)	For plane *OAS*, for its normal vector we have $\overrightarrow{OA}\times\overrightarrow{OS}=\begin{pmatrix}6\\-8\\0\end{pmatrix}\times\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}-16\\-12\\-8\end{pmatrix}=-4\begin{pmatrix}4\\3\\2\end{pmatrix}$ We take $\mathbf{n}=\begin{pmatrix}4\\3\\2\end{pmatrix}$ Plane *OAS* contains the origin. Plane *OAS*: $\mathbf{r}\cdot\begin{pmatrix}4\\3\\2\end{pmatrix}=0$ ---- (1) Let l_{FN} be the line that passes through *F* and *N* $l_{FN}:\mathbf{r}=\begin{pmatrix}-2.8\\-3.6\\0\end{pmatrix}+\beta\begin{pmatrix}4\\3\\2\end{pmatrix},\ \beta\in\mathbb{R}$ ---- (2) Sub (2) into (1): $\left[\begin{pmatrix}-2.8\\-3.6\\0\end{pmatrix}+\beta\begin{pmatrix}4\\3\\2\end{pmatrix}\right]\cdot\begin{pmatrix}4\\3\\2\end{pmatrix}=0$ $-22+29\beta=0$ $\beta=\frac{22}{29}$ Sub $\beta=\frac{22}{29}$ into eqn (2): $\overrightarrow{ON}=\begin{pmatrix}-2.8\\-3.6\\0\end{pmatrix}+\frac{22}{9}\begin{pmatrix}4\\3\\2\end{pmatrix}$	Some students failed to realize that they have to carry out the cross product $\overrightarrow{OA}\times\overrightarrow{OS}$ to find the normal of plane *OAS*. Some mistook $\overrightarrow{OC}$, $\overrightarrow{OF}$ as the required normal. A lot of algebraic and arithmetic mistakes.

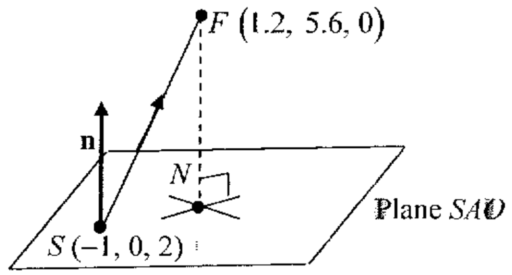
$|\overrightarrow{FN}|$

$= |\overrightarrow{ON} - \overrightarrow{OF}|$

$$= \left| \begin{pmatrix} -2.8 \\ -3.6 \\ 0 \end{pmatrix} + \frac{22}{29}\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2.8 \\ -3.6 \\ 0 \end{pmatrix} \right|$$

$$= \frac{22}{29}\left|\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}\right| = \frac{22\sqrt{29}}{29} = \frac{22}{\sqrt{29}} \text{ units}$$

Alternatively



Based on the given information, we note that the point N is the foot of the perpendicular from point F to plane OAS.

For plane OAS, for its normal vector we have

$$\overrightarrow{OA} \times \overrightarrow{OS} = \begin{pmatrix} 6 \\ -8 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -16 \\ -12 \\ -8 \end{pmatrix} = -4\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

We take $\mathbf{n} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

We next note that the distance $|\overrightarrow{FN}| = |\overrightarrow{SF} \cdot \text{n}|$.

So, we $|\overrightarrow{FN}| = |\overrightarrow{SF} \cdot \text{n}| = |(\overrightarrow{OF} - \overrightarrow{OS}) \cdot \text{n}|$

$$= \left| \left(\begin{pmatrix} -2.8 \\ -3.6 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \right| \div \sqrt{4^2 + 3^2 + 2^2}$$

	$= \left\| \begin{pmatrix} -1.8 \\ -3.6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \right\| \div \sqrt{4^2+3^2+2^2}$ $= \dfrac{\left\|-4(1.8)-3(3.6)-2(2)\right\|}{\sqrt{29}}$ $= \dfrac{22}{\sqrt{29}} = 4.085297 \approx 4.09$ units	
(iv)	ΔSFN is a right angle Δ with $\angle FNS = 90^o$ $\cos \angle SFN = \dfrac{\left\|\overrightarrow{FN}\right\|}{\left\|\overrightarrow{FS}\right\|}$ $= \dfrac{22}{\sqrt{29}} \div \sqrt{1.8^2+3.6^2+2^2}$ $= \dfrac{22}{\sqrt{29}\sqrt{20.2}}$ Hence $\angle SFN = \cos^{-1}\left(\dfrac{22}{\sqrt{29}\sqrt{20.2}}\right) \approx 24.6^o$ Alternatively The required angle is $\angle SFN$ Let θ be the required angle, then θ is the angle between $\overrightarrow{FS}$ and $\overrightarrow{FN}$. From (iii), $\overrightarrow{FN} = \dfrac{22}{29}\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ $\overrightarrow{FS} = \overrightarrow{OS} - \overrightarrow{OF} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -2.8 \\ -3.6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 3.6 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1.8 \\ 3.6 \\ 2 \end{pmatrix} = \left\|\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1.8 \\ 3.6 \\ 2 \end{pmatrix}\right\| \cos\theta$	A number of students failed to interpret the context and mistook that the question requires them to find the angle between the line *FS* and plane *OAS*.

	$22=\left(\sqrt{29}\right)\left(\sqrt{20.2}\right)\cos\theta$ $\theta=24.64^{o}\approx 24.6^{o}$		

12	Solution [12] Differential Equations	
(i)	$\frac{dv}{dt} = -\mu v - \lambda v^3$ --(1) $z = \frac{1}{v^2}$ assuming $v \neq 0$ --(2) $\frac{dz}{dt} = \frac{dz}{dv} \cdot \frac{dv}{dt}$ $= \frac{-2}{v^3}\frac{dv}{dt}$ $= \left(\frac{-2}{v^3}\right)\left(-\mu v - \lambda v^3\right)$ --(3) $= \frac{2\mu}{v^2} + 2\lambda$ $= 2\mu z + 2\lambda$ (Shown) $\int \frac{1}{\mu z + \lambda}\,dz = \int 2\,dt$ $\frac{1}{\mu}\ln\lvert \mu z + \lambda \rvert = 2t + c$ $\mu z + \lambda = \pm e^{2\mu t + \mu c}$ $z = -\frac{\lambda}{\mu} \pm \frac{1}{\mu} e^{2\mu t + \mu c}$ $= B + Ae^{2\mu t}$ where $A = \pm\frac{1}{\mu}e^{\mu c}$ and $B = -\frac{\lambda}{\mu}$ $v^2 = \frac{1}{B + Ae^{2\mu t}}$ $v = \frac{1}{\sqrt{B + Ae^{2\mu t}}}$	Most students are able to prove that $\frac{dz}{dt} = 2\mu z + 2\lambda$. However some students do not realize that $\frac{dz}{dt} = 2\mu z + 2\lambda$ is supposedly a more manageable DE to solve than the original DE $\frac{dv}{dt} = -\mu v - \lambda v^3$. Many students failed to show clearly how the modulus sign in $\frac{1}{\mu}\ln\lvert \mu z + \lambda \rvert = 2t + c$ is removed by stating that $A = \pm\frac{1}{\mu}e^{\mu c}$
(ii)	As $t \to \infty$, $v \to 0$	
(iii)	If $\mu = 0.1$, $\lambda = 0.08$ $v = \frac{1}{\sqrt{B + Ae^{2\mu t}}}$ where $B = -\frac{\lambda}{\mu}$, $v = \frac{1}{\sqrt{-\frac{0.08}{0.1} + Ae^{(0.2)t}}}$ $v = \frac{1}{\sqrt{Ae^{(0.2)t} - 0.8}}$	Generally, the students make appropriate substitutions using the provided values

	Sub $v=1$, $t=3$, $1=\dfrac{1}{\sqrt{Ae^{0.6}-0.8}}$ $A=1.8e^{-0.6}$ $v=\dfrac{1}{\sqrt{1.8e^{0.2t-0.6}-0.8}}$	Many students failed to adequately simplify the final expression of v, leaving it in the form $v=\sqrt{\dfrac{0.1}{18e^{0.2t-0.6}-8}}$

2021 H2 Maths Prelim P2

1	Solution [4] System of Linear Equations	
	Method 1 $a\left(1-\sqrt{3}\text{ i}\right)^3+b\left(1-\sqrt{3}\text{ i}\right)^2+c\left(1-\sqrt{3}\text{ i}\right)+d=0$ $a(-8)+b\left(1-2\sqrt{3}\text{ i}-3\right)+c\left(1-\sqrt{3}\text{ i}\right)+d=0$ $a(-8)+b\left(-2-2\sqrt{3}\text{ i}\right)+c\left(1-\sqrt{3}\text{ i}\right)+d=0$ Thus, we have the 2 equations: $2b-c-d+8a=0$ ------(1) $2b+c=0$ ---------(2) Also, since $\frac{3}{2}$ is a root for $az^3+bz^2+cz+d=0$, $a\left(\frac{3}{2}\right)^3+b\left(\frac{3}{2}\right)^2+c\left(\frac{3}{2}\right)+d=0$ $\frac{27}{8}a+\frac{9}{4}b+\frac{3}{2}c+d=0$ $\Rightarrow 18b+12c+8d+27a=0$ -----(3) Using GC $b=-\frac{7}{2}a$, $\text{c}=7a$ and $\text{d}=-6a$	Quite averagely done. Many students did not read the question properly as you were tasked to express the answers in terms of a. Many also concluded immediately that $1+\sqrt{3}\text{ i}$ is a root and have gotten stuck there. Some students did not substitute the roots as well into the equation.
	Method 2 Consider $az^3+bz^2+cz+d=0$. Divide throughout by a, then $z^3+b'z^2+c'z+d'=0$ where $b'=\frac{b}{a}$, $c'=\frac{c}{a}$, $d'=\frac{d}{a}$ Since $1-\sqrt{3}\text{ i}$ is a root for the eqn $z^3+b'z^2+c'z+d'=0$, $\left(1-\sqrt{3}\text{ i}\right)^3+b'\left(1-\sqrt{3}\text{ i}\right)^2+c'\left(1-\sqrt{3}\text{ i}\right)+d'=0$ $(-8)+b'\left(1-2\sqrt{3}\text{ i}-3\right)+c'\left(1-\sqrt{3}\text{ i}\right)+d'=0$ $\left(-8-2b'+c'+d'\right)+\left(-2b'-c'\right)\sqrt{3}\text{ i}=0$ Thus, we have the 2 equations: $2b'-c'-d'=-8$ ------(1) $2b'+c'=0$ ---------(2) Also, since $\frac{3}{2}$ is a root for $z^3+b'z^2+c'z+d'=0$,	

	$\left(\frac{3}{2}\right)^3+b'\left(\frac{3}{2}\right)^2+c'\left(\frac{3}{2}\right)+d'=0$ $\frac{27}{8}+\frac{9}{4}b'+\frac{3}{2}c'+d'=0$ $\Rightarrow 18b'+12c'+8d'=-27$ ------(3) Using GC to solve the above equations (1), (2), (3), we have $b'=-\frac{7}{2}$, c' = 7 and d' = −6. $b=-\frac{7}{2}a$, c = $7a$ and d = $-6a$	

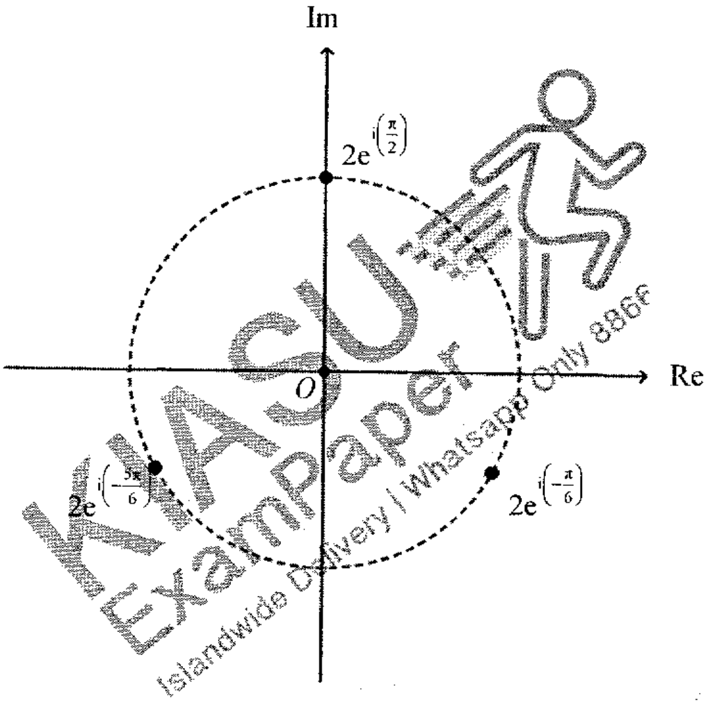
2	Solution [6] Summation with method of difference	
(i)	$\text{LHS}=\sin\left(p+\frac{1}{2}\right)\theta-\sin\left(p-\frac{1}{2}\right)\theta$ $=\left[\sin p\theta\cos\frac{1}{2}\theta+\cos p\theta\sin\frac{1}{2}\theta\right]-\left[\sin p\theta\cos\frac{1}{2}\theta-\cos p\theta\sin\frac{1}{2}\theta\right]$ $=2\cos p\theta\sin\frac{1}{2}\theta=\text{RHS}$ (shown)	Quite well done, except for quite a number who has forgotten the formula from Secondary 4.
(ii)	$\sum_{p=1}^{n}\cos p\theta$ $=\sum_{p=1}^{n}\frac{\sin\left(p+\frac{1}{2}\right)\theta-\sin\left(p-\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$ $=\frac{1}{2\sin\frac{1}{2}\theta}\sum_{p=1}^{n}\left[\sin\left(p+\frac{1}{2}\right)\theta-\sin\left(p-\frac{1}{2}\right)\theta\right]$ $=\frac{1}{2\sin\frac{1}{2}\theta}\left[\sin\frac{3}{2}\theta-\sin\frac{1}{2}\theta\right.$ $+\sin\frac{5}{2}\theta-\sin\frac{3}{2}\theta$ $+\sin\frac{7}{2}\theta-\sin\frac{5}{2}\theta$ $+\ldots$ $+\sin\left(n-\frac{1}{2}\right)\theta-\sin\left(n-\frac{3}{2}\right)\theta$ $\left.+\sin\left(n+\frac{1}{2}\right)\theta-\sin\left(n-\frac{1}{2}\right)\theta\right]$ $=\frac{1}{2\sin\frac{1}{2}\theta}\left[\sin\left(n+\frac{1}{2}\right)\theta-\sin\frac{1}{2}\theta\right]$ $=\frac{\operatorname{cosec}\frac{1}{2}\theta}{2}\left[\sin\left(n+\frac{1}{2}\right)\theta-\sin\frac{1}{2}\theta\right]$ (Shown)	Quite badly done. This is a very standard method of difference question where many students were unable to identify. Some were also unable to identify the need to factorize out $\frac{1}{2\sin\frac{1}{2}\theta}$ to match the expression given in the previous parts.

(iii)	$\sum_{p=5}^{32}\cos(p-2)\theta$ $=\sum_{r=3}^{30}\cos r\theta \qquad (\text{replace } p \text{ by } r+2)$ $=\sum_{r=1}^{30}\cos r\theta-\cos\theta-\cos 2\theta$ $=\frac{1}{2\sin\frac{1}{2}\theta}\left[\sin\left(30+\frac{1}{2}\right)\theta-\sin\frac{1}{2}\theta\right]-\cos\theta-\cos 2\theta$ When $\theta=\frac{\pi}{2}$, $\sum_{p=5}^{32}\cos\left[(p-2)\left(\frac{\pi}{2}\right)\right]$ $=\frac{1}{2\sin\frac{\pi}{2}}\left[\sin\left(15\pi+\frac{\pi}{4}\right)-\sin\frac{\pi}{4}\right]-\cos\frac{\pi}{2}-\cos\frac{2\pi}{2}$ $=\frac{\sqrt{2}}{2}\left[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right]-0-(-1)=-1+1=0$	Very badly done. Despite going through these types of questions in tutorials, many had trouble identifying what to replace p with. Many students also left this empty; it will be good to expose yourself more to limits-changing types of questions like this.

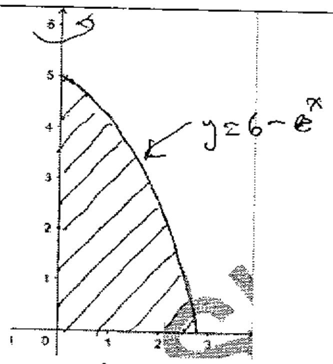
3	Solution [8] Differentiation, tangent/normal	
(a)	Note that $4y^2-2(x-1)y=-1$ is equivalent to $4y^2+2y(1-x)=-1$ $y^2+xy=-1$ $\xrightarrow{\text{A: replace } y \text{ with } 2y}$ $(2y)^2+x(2y)=-1$ $\Rightarrow 4y^2+2xy=-1$ $\xrightarrow{\text{B: replace } x \text{ with } -x}$ $4y^2+2(-x)y=-1$ $\Rightarrow 4y^2-2xy=-1$ $\xrightarrow{\text{C: replace } x \text{ with } x-1}$ $4y^2-2(x-1)y=-1$ A: Scale the graph parallel to the y-axis by factor $\frac{1}{2}$ B: Reflect the graph about the y-axis C: Translate the graph 1 unit in the positive x-direction OR $y^2+xy=-1$ $\xrightarrow{\text{A: replace } y \text{ with } 2y}$ $(2y)^2+x(2y)=-1$ $\Rightarrow 4y^2+2xy=-1$ $\xrightarrow{\text{B: replace } x \text{ with } x+1}$ $4y^2+2(x+1)y=-1$ $\xrightarrow{\text{C: replace } x \text{ with } -x}$ $2y^2+2((-x)+1)y=-1$ $\Rightarrow 2y^2-2(x-1)y=-1$ A: Scale the graph parallel to the y-axis by factor $\frac{1}{2}$ B:Translate the graph 1 unit in the negative x-direction C: Reflect the graph about the y-axis	Generally ok. Students need to be careful with the order of transformations eg some perform translation of 1 unit in positive x-direction followed by a reflection about the y-axis which resulted in the wrong expression $(-x-1)=-(x+1)$ instead of the required $-(x-1)$. To replace y with $-2y$, the correct step-by-step presentation should be a scaling parallel to the y-axis with scale factor $\frac{1}{2}$ followed by a reflection about the x-axis and not just a scaling parallel to the y-axis with scale factor $-\frac{1}{2}$. Students should use the correct terms like translation, scaling and reflection in the description of transformation instead of general words like flip, shift, invert, mirror, stretch, extend etc.

(b) (i)	$y^2+xy=-1$ Differentiating both sides with respect to x, $2y\frac{dy}{dx}+x\frac{dy}{dx}+y=0$ $\frac{dy}{dx}(2y+x)=-y$ $\frac{dy}{dx}=\frac{-y}{2y+x}$	Small number of students performed the implicit differentiation steps without equating the LHS to 0 in their derivation for expression for $\frac{dy}{dx}$.
(b) (ii)	When $y=a, a^2+ax=-1$ $x=-\frac{a^2+1}{a}$ $\frac{dy}{dx}=\frac{-a}{2a-\frac{a^2+1}{a}}$ $=\frac{-a^2}{a^2-1}=\frac{a^2}{1-a^2}$ Equation of tangent: $y-a=\frac{a^2}{1-a^2}\left(x-\left(-\frac{a^2+1}{a}\right)\right)$ $y=\frac{a^2}{1-a^2}x+\left(a+\frac{a(a^2+1)}{1-a^2}\right)$ $=\frac{a^2}{1-a^2}x+\frac{a(1-a^2)+a(1+a^2)}{1-a^2}$ $=\frac{a^2}{1-a^2}x+\frac{2a}{1-a^2}$	Generally no problem in finding the x coordinates and the expression of $\frac{dy}{dx}$ in terms of a. However, some students did not substitute x in terms of a for $\frac{dy}{dx}=\frac{-y}{2y+x}$ and resulting in two "x"s in the equation for the tangent as needed.
(b) (iii)	As $a\to 1, x\to -2$ and $\frac{dy}{dx}\to\infty$. The tangent becomes the vertical line $x=-2$	Common presentation errors: tangent = $-\infty$ tangent tends to infinity tangent tends to y-axis tangent approaches asymptote

4	Solution [9] Complex Numbers	
(i)	$8e^{i\left(2k\pi-\frac{\pi}{2}\right)}=8e^{i\left(-\frac{\pi}{2}\right)}e^{i(2k\pi)}$ $=8e^{i\left(-\frac{\pi}{2}\right)}\left(e^{i(2\pi)}\right)^k$ $=8e^{i\left(-\frac{\pi}{2}\right)}(1)^k$ $=8\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$ $=8(0+i(-1))$ $=-8i$ (verified)	Common error in verifying: $e^{i\theta}=\cos\theta-i\sin\theta$ Many student did not mention that $\left(e^{i(2\pi)}\right)^k=(1)^k$ and merely just stated that $8e^{i\left(2k\pi-\frac{\pi}{2}\right)}=8e^{i\left(-\frac{\pi}{2}\right)}$
(ii)	$-\pi<\frac{1}{3}\left(2k\pi-\frac{\pi}{2}\right)\le\pi$ $-3\pi+\frac{\pi}{2}<2k\pi\le3\pi+\frac{\pi}{2}$ $-\frac{5}{4}<k\le\frac{7}{4}$ $\Rightarrow k=-1,0,1$	Many students verified value of k instead of showing the range of possible values of k. Some just expressed $k=\frac{1}{4}+\frac{3\theta}{2\pi}$ and did not provide further explanations.
(iii)	By the Fundamental theorem of Algebra, the cubic equation will have 3 roots.	Some students stated that the degree of the given equation or the highest power of w is 3
(iv)	$w^3=-8i$ $w^3=8e^{i\left(2k\pi-\frac{\pi}{2}\right)}$ from (i) $w=8^{\frac{1}{3}}e^{i\left(2k\pi-\frac{\pi}{2}\right)\left(\frac{1}{3}\right)}$ $=2e^{i\left(\frac{2k\pi}{3}-\frac{\pi}{6}\right)}$ Since, $-\pi<\frac{1}{3}\left(2k\pi-\frac{\pi}{2}\right)\le\pi$, using part (ii) result, $k=-1,0,1$ when $k=-1$, $w=2e^{i\left(-\frac{5\pi}{6}\right)}$ when $k=0$, $w=2e^{i\left(-\frac{\pi}{6}\right)}$ when $k=1$, $w=2e^{i\left(\frac{3\pi}{6}\right)}=2e^{i\left(\frac{\pi}{2}\right)}$	Not well done. Many students did not realise that the result of part (ii) can be used for this part. They did not notice that $w=8^{\frac{1}{3}}e^{i\left(2k\pi-\frac{\pi}{2}\right)\left(\frac{1}{3}\right)}$ and mistakes involved were either modulus of w is still 8 or the argument of w are wrong value like $2(-1)\pi-\frac{\pi}{2}=-\frac{5}{2}\pi$, $\frac{\pi}{2}$ or $2(1)\pi-\frac{\pi}{2}=\frac{3}{2}\pi$ Some students attempted to find the roots of the equation in algebraic way

	 	by letting $w=a+bi$ and formed equation involving unknowns a and b. However, most of them only managed to find one or two of the 3 roots using this method. Some students did not realise that the question requested for roots in the form $re^{i\theta}$ and left their answers all in $a+bi$ form. Generally, for students who managed to find the 3 correct roots, there was no problem in the drawing of the needed Argand diagram for the 3 roots.
(v)	From diagram, The 3 roots lie on $\lvert z\rvert=2$ Therefore, $a=0$ and $r=2$	Very few students showed correct understanding of the geometrical aspect of the equation $\lvert z-a\rvert=r$. Majority of the students did not realise that the centre of circle was then the origin and thus a should be 0.

5	Solution [12] Integration & Application	
(a)	$\int \ln(a-x)\,dx = \int (1)(\ln(a-x))\,dx$ Let $u'=1 \Rightarrow u=x$ $v=\ln(a-x) \Rightarrow v'=\dfrac{-1}{a-x}$ $\int \ln(a-x)\,dx = \int (1)(\ln(a-x))\,dx$ $= x\ln(a-x) - \int \left(\dfrac{-x}{a-x}\right) dx$ $= x\ln(a-x) - \int \left(\dfrac{a-x-a}{a-x}\right) dx$ $= x\ln(a-x) - x + a\int \dfrac{1}{a-x}\,dx$ $= x\ln(a-x) - x - a\ln(a-x) + c$ $= (x-a)\ln(a-x) - x + c$ (Shown)	Varying standards among all students. Most students were able to first apply 'Integration by Parts' method in finding $\int \ln(a-x)\,dx$ and $\int (\ln(a-x))^2\,dx$. However, many of them did not realise the useful step in rewriting $\dfrac{x}{a-x} = -1 + \dfrac{a}{a-x}$ in the 2nd part of the integrating process of both integrals as shown in solution.
	$\int (\ln(a-x))^2\,dx = \int (1)(\ln(a-x))^2\,dx$ Let $u'=1 \Rightarrow u=x$ $v=(\ln(a-x))^2 \Rightarrow v'=\dfrac{-2}{a-x}\ln(a-x)$ $\int (\ln(a-x))^2\,dx$ $= \int (1)(\ln(a-x))^2\,dx$ $= x(\ln(a-x))^2 - \int (x)\left(\dfrac{-2}{a-x}\ln(a-x)\right) dx$ $= x(\ln(a-x))^2 + 2\int \left(-1+\dfrac{a}{a-x}\right)\ln(a-x)\,dx$ $= x(\ln(a-x))^2 - 2a\int (\ln(a-x))\left(\dfrac{-1}{a-x}\right) dx$ $-2\int \ln(a-x)\,dx$ $= x(\ln(a-x))^2 - a(\ln(a-x))^2$ $-2(x-a)\ln(a-x) + 2x + c$ $= (x-a)(\ln(a-x))^2 - 2(x-a)\ln(a-x) + 2x + c$	For $\int (\ln(a-x))^2\,dx$, common errors include: $\int (\ln(a-x))^2\,dx$ $= 2\int \ln(a-x)\,dx$; $\int (\ln(a-x))^2\,dx$ $= \left(\int \ln(a-x)\,dx\right)^2$ which severely affected answer for the later part (b)(i). Some students also got mixed up with expressions involving $(x-a)$ and $(a-x)$ in the integrating processes.

(b) (i)	 $V = \pi\int_0^5 x^2\ dy$ $= \pi\int_0^5 (\ln(6-y))^2\ dy$ $= \pi\left[\left((y-6)(\ln(6-y))^2 - 2(y-6)\ln(6-y) + 2y\right)\right]_0^5$ $= \pi\left[(-1)(0)^2 - 2(-1)(0) + 2(5)\right]$ $-\pi\left[(-6)(\ln 6)^2 - 2(-6)\ln 6 + 0\right]$ $= \pi\left(10 - 12(\ln 6) + 6(\ln 6)^2\right)$	No marks for students who apply wrongly volume = $\pi\int y^2\ dx$ as the volume generated is about the y-axis. Other common issue with this part involves careless arithmetical mistakes with the definite integral and wrongly using the result for $\int \ln(a-x)\,dx$ instead or wrong result for $\int(\ln(a-x))^2\ dx$ in part(a). There were also students who quoted other wrong formula like $V = \pi\int_0^5 x\ dy$ or $V = 2\pi\int_0^5 x^2\ dy$.
(b) (ii)	$A = 2\int_0^5 x\ dy$ $= 2\int_0^5 \ln(6-y)\ dy$ $= 2\left[(y-6)\ln(6-y) - y\right]_0^5$ $= 2\left[(-1)\ln(1) - 5\right] - 2\left[-6\ln(6) - 0\right]$ $= 12\ln 6 - 10$ Alternatively $y = 6 - e^x$ When $y = 0$, $6 - e^x = 0 \Rightarrow x = \ln 6$ When $y = 5$, $6 - e^x = 5 \Rightarrow e^x = 1 \Rightarrow x = 0$ $\int_0^{\ln 6} y\ dx$ $= \int_0^{\ln 6} 6 - e^x\ dx$ $= [6x]_0^{\ln 6} - \left[e^x\right]_0^{\ln 6}$ $= 6\ln 6 - 5$ $A = 2 \times [6\ln 6 - 5]$ $A = 12\ln 6 - 10$	Many students only computed $A = \int_0^5 \ln(6-y)\ dy$ or $\int_0^{\ln 6} 6 - e^x\ dx$ for this part. There were also some who did not give the exact answer as needed.

6	Solution [6] P & C	
(i)	Number of passcodes $=\binom{10}{5}(5!)=30240$	Many wrongly counted there to be 9 digits from 0 to 9.
(ii)	Let A, B, C be integers such that $0\le A\le 9$, $0\le B\le 9$ and $0\le C\le 9$. Case 1: Permutations of digits in the string AABBC Number of passcodes $=\binom{10}{3}\binom{3}{2}\frac{5!}{2!2!}=10800$ Case 2: Permutations of digits in the string AABBB Number of passcodes $=\binom{10}{2}\binom{2}{1}\frac{5!}{3!2!}=900$ Total number of passcodes $=10800+900=11700$	Many failed to split up into the correct cases. Many did not <u>DESCRIBE THE cases clearly</u>
(iii)	$\mathrm{P}(\text{passcode contains repeated digit(s)})$ $=\dfrac{\mathrm{n}(\text{passcodes with repeated digit(s)})}{\mathrm{n}(5\text{ digit passcode})}$ $=\dfrac{10^5-30240}{10^5}$ $=\dfrac{6976}{10000}=0.6976$	Most see that part (iii) is related to part (i), and are able to apply

<table>
<tr><td>7</td><td>Solution [8] DRV</td><td></td></tr>
<tr><td>(a)</td><td>

$$P(X=2)=\frac{\binom{3}{2}\binom{9}{3}}{\binom{12}{5}}=\frac{7}{22}\text{ (shown)}$$

<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$P(X=x)$</td><td>$\frac{\binom{9}{5}}{\binom{12}{5}}=\frac{7}{44}$</td><td>$\frac{\binom{3}{1}\binom{9}{4}}{\binom{12}{5}}=\frac{21}{44}$</td><td>$\frac{7}{22}$</td><td>$\frac{\binom{3}{3}\binom{9}{2}}{\binom{12}{5}}=\frac{1}{22}$</td></tr>
</table>

</td><td>Students were generally able to score well for this question.

However, many students used more complicated methods than were required.

Eg: $\frac{3}{12}\frac{2}{11}\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{5!}{2!3!}+\frac{3}{12}\frac{2}{11}\frac{5}{10}\frac{4}{9}\frac{4}{8}\frac{5!}{2!2!}+\frac{3}{12}\frac{2}{11}\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{5!}{2!2!}+\frac{3}{12}\frac{2}{11}\frac{4}{10}\frac{3}{9}\frac{2}{8}\frac{5!}{2!3!}$</td></tr>
<tr><td>(b)</td><td>

$$E(X)=0\left(\frac{7}{44}\right)+1\left(\frac{21}{44}\right)+2\left(\frac{7}{22}\right)+3\left(\frac{1}{22}\right)=1.25$$

$$E(X^2)=0^2\left(\frac{7}{44}\right)+1^2\left(\frac{21}{44}\right)+2^2\left(\frac{7}{22}\right)+3^2\left(\frac{1}{22}\right)=\frac{95}{44}$$

$$\text{Var}(X)=E(X^2)-[E(X)]^2=0.596591\approx 0.597\text{ (to 3sf)}$$

</td><td>Generally well done if students managed to generate the PDF in (a).

Some students forgot the formula for Var(X), and instead found $E(X^2)$</td></tr>
<tr><td>(c)
(iii)</td><td>

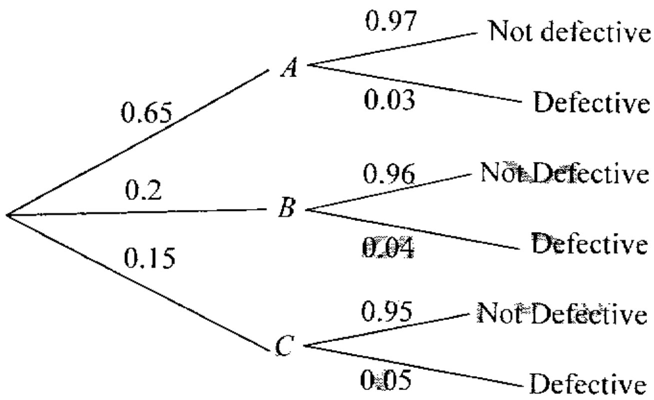
$E(2X-3)=2(1.25)-3=-0.5$,

The value of $2X-3$ gives the profit for each game.

Thus, the player will be losing $0.50 for each round on average in the long run.

</td><td>Generally well done.

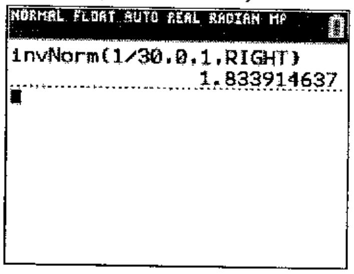
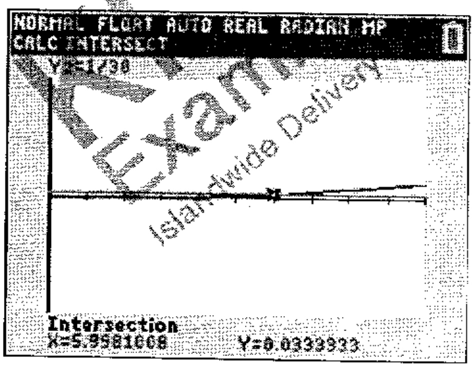
Students should try to be precise and succinct in their description.</td></tr>
</table>

8	Solution [8] Probability	
(i)		Generally well done. A small number put $P(A \cap D)$ instead of $P(D \mid A)$ on the second column of branches.
(ii)	P(a sprayer has manufacturing defect) $= 0.65 \times 0.03 + 0.2 \times 0.04 + 0.15 \times 0.05$ $= 0.035$ P(1 out of 2 sprayers has manufacturing defect) $= \binom{2}{1} \times 0.035 \times (1 - 0.035)$ $= 0.06755$	Many students tried to use the tree diagram to do all of this question. Some were able to do this, but many lacked the accuracy to do so. Common errors included failing to account for the order of selection, and failing to account for the event in which the same type is chosen twice.
	<u>Alternative</u> (*Not advised, but this was a common method*) P(sprayer has defect) $= P(A \cap A_D) + P(A \cap B_D) + P(A \cap C_D)$ $+ P(B \cap A_D) + P(B \cap B_D) + P(B \cap C_D)$ $+ P(C \cap A_D) + P(C \cap B_D) + P(C \cap C_D)$ $= \begin{bmatrix} (0.65)(0.97)(0.65)(0.03) + (0.65)(0.97)(0.2)(0.04) + (0.65)(0.97)(0.15)(0.05) \\ +(0.2)(0.96)(0.65)(0.03) + (0.2)(0.96)(0.2)(0.04) + (0.2)(0.96)(0.15)(0.05) \\ (0.15)(0.95)(0.65)(0.03) + (0.15)(0.95)(0.2)(0.04) + (0.15)(0.95)(0.15)(0.05) \end{bmatrix} \times 2$ $= \frac{1351}{20\ 000}$	

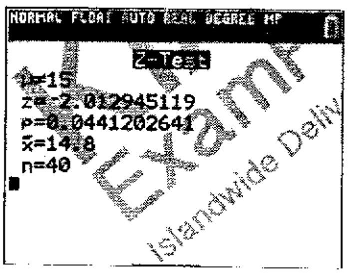
(iii)	$\text{P}(2 \text{ Type C sprayers} \mid \text{Exactly 1 sprayer has defect})$ $= \dfrac{\text{P}(2 \text{ Type C sprayers and exactly 1 sprayer has defect})}{\text{P}(\text{Exactly 1 sprayer has defect})}$ $= \dfrac{\begin{bmatrix}\text{P}(CCA, A \text{ defective}) + \text{P}(CCA, C \text{ defective}) \\ +\text{P}(CCB, B \text{ defective}) + \text{P}(CCB, C \text{ defective})\end{bmatrix}}{\text{P}(\text{Exactly 1 sprayer has defect})}$ $= \dfrac{\begin{bmatrix}(0.15\times0.95)^2(0.65\times0.03)\times\left(\frac{3!}{2!}\right) \\ +(0.15\times0.05)(0.15\times0.95)(0.65\times0.97)\times3! \\ +(0.15\times0.95)^2(0.2\times0.04)\times\left(\frac{3!}{2!}\right) \\ +(0.15\times0.05)(0.15\times0.95)(0.2\times0.96)\times3!\end{bmatrix}}{\binom{3}{1}(0.035)(1-0.035)^2}$ $=0.0711$	This question was not well done. Most students were able to identify that there was a conditional probability. Many students were not able to correctly identify all cases. Many students did not have proper permutations of the cases. Many students were not able to extrapolate the case in (ii) into the denominator.

9	Solution [8] Binomial Distribution	
(i)	The event that John manages to answer a multiple-choice question correctly is independent of him answering any other multiple-choice questions correctly. OR The probability of John answering a question correctly is a constant.	Generally well done, although students should not that **events** are independent, not probabilities.
(ii)	Let Y be the total score of John in the first 4 rounds. Then $Y \sim B(40, 0.85)$ So $\text{P}(Y>36) = 1-\text{P}(Y\le36)$ $= 1-0.8698312384$ $= 0.130$ (3 s.f.)	Generally well done. Many students improperly presented the Binomial distribution. E.g. $X \sim B(40, 0.85)$ (which is not appropriate as X is defined in the question) Some students had $\text{P}(Y>36) = 1-\text{P}(Y\le35)$ Some students incorrectly attempted to apply CLT.

(iii)	$X \sim B(10,\ 0.85)$ Required probability = $P(X_1 \le 9) \times P(X_2 \le 9)$ $= (0.8031255956)^2$ $= 0.645$ (3 s.f.)	Generally well done.
(iv)	For $X \sim B(10,\ 0.85)$, mean = $np = 10 \times 0.85 = 8.5$ variance $= np(1-p) = 10 \times 0.85 \times (1-0.85) = 1.275$ Then for 50 shooting practice rounds, $\overline{X} = \dfrac{X_1 + X_2 + X_3 + \ldots\ldots + X_{50}}{50}$ and $\overline{X} \sim N\left(8.5, \dfrac{1.275}{50}\right)$ approximately by Central Limit Thm Hence $P(\overline{X} \ge 8.8) = 0.0301$ (3 s.f.)	Not well done. Many students were not able to apply CLT. There were many poor attempts at calculating the variance.
	<u>Alternative</u> Let $T = X_1 + X_2 + \ldots + X_{50} \sim B(500, 0.85)$ $P(\overline{X} \ge 8.8) = P(T \ge 440)$ $= 1 - P(T \le 439)$ $= 0.319$	
	<u>Alternative:</u> Let $T = X_1 + X_2 + \ldots + X_{50}$ Then $E(T) = 425$, $Var(T) = 63.75$ $T \sim N(425, 63.75)$ approx by CLT $\because$ $n = 50$ is large $P(\overline{X} \ge 8.8) = P(T \ge 440) = 0.0301$	

10	Solution [9] Normal Distribution	
(i)	Let E : English marks for a P6 student in the district in the exam. Then $E \sim \mathrm{N}\left(76,\ 5^2\right)$. Let M : Maths marks for a P6 student in the district in the exam. Then $M \sim \mathrm{N}\left(74,\ \sigma^2\right)$. Given $\mathrm{P}(M \geq 85) = \frac{1}{30}$, we have $\mathrm{P}\left(Z \geq \frac{85-74}{\sigma}\right) = \frac{1}{30}$.  Then, $\frac{11}{\sigma} = 1.833914637 \Rightarrow \sigma = 5.99809815$ Thus, $\sigma = 6$ (corrected to nearest whole number) OR Plot $Y1 = P\left(Z \geq \frac{11}{x}\right)$ $Y2 = \frac{1}{30}$  The x-coordinate of the point of intersection is $= 5.9981 \approx 6$	Most students are able to use standardization and InvNorm to formulate the equation. However, there are some who are still not sure standardization and write $\mathrm{P}\left(Z \geq \frac{85-74}{\sigma^2}\right) = \frac{1}{30}$ instead. There are some students who use the OR method but did not show the working/graph to illustrate how they arrived at the answer.

(ii)	$E \sim \mathrm{N}(76,\ 25)$ $M \sim \mathrm{N}(74,\ 36)$ $2E+M \sim \mathrm{N}\left(2\times 76+74,\ 2^2\times 5^2+6^2\right)$ $\Rightarrow 2E+M \sim \mathrm{N}(226,\ 136)$ Then the required probability $=\dfrac{P(E>85)\times P(M>85)}{P(2E+M\geq 250)}$ $=\dfrac{0.0359302655\times 0.0333764484}{0.0197958145}$ $=0.0605797075$ $=0.0606$ (3 s.f.)	Quite a number of students formulate the expression of the probability wrongly: $\dfrac{0.0359302655\times 0.0333764484\times}{0.0197958145}0.0197958145$ They tend to multiply the numerator with 0.0197.
(iii)	We first note that: $X=E_1+E_2+E_3+E_4+E_5 \sim \mathrm{N}\left(5\times 76,\ 5\times 5^2\right)$ $Y=M_1+M_2+M_3+M_4+M_5 \sim \mathrm{N}\left(5\times 74,\ 5\times 6^2\right)$ Thus, $X-Y \sim \mathrm{N}(10,\ 305)$. Next, we have: $\mathrm{P}(\lvert X-Y\rvert>30)$ $=\mathrm{P}(X-Y<-30)+\mathrm{P}(X-Y>30)$ $=0.0109992317+0.126063901$ $=0.137$ (3 s.f.)	This part proved to be challenging as quite a number of students either consider $\mathrm{P}(X-Y<-30)$ only or $\mathrm{P}(-30<X-Y<30)$
(iv)	$E \sim \mathrm{N}(76,\ 25)$ $\mathrm{P}(E>80)=0.2118553337$ Let A denote the number of students out of n students who score more than 80 marks in the examination. $A \sim \mathrm{B}(n,\ 0.2118553337)$ $\mathrm{P}(A>4)>0.15$ $1-\mathrm{P}(A\leq 4)>0.15$ $\mathrm{P}(A\leq 4)<0.85$ Using GC, When $n=13$, $\mathrm{P}(A\leq 4)=0.8792$ When $n=14$, $\mathrm{P}(A\leq 4)=0.8434$ The least value of n is 14.	This part is quite well done by most students except the last part when they interpret $\mathrm{P}(A>4)>0.15$ as $\mathrm{P}(A>4)>0.15$ $1-\mathrm{P}(A\leq 3)>0.15$

11	Solution [14] Hypothesis Testing	
(i)	1. The production manager can prepare a list of all the workers arranged according to alphabetical order of their name. and assigning each worker a number 2. Then the manager uses a random number generator to generate 40 numbers and pick out the corresponding 40 workers.	Most of the students got the idea of randomness and the use of random number generator.
(ii)	Unbiased estimate for population mean is $\bar{t}=\frac{\sum(t-15)}{n}+15=\frac{-8}{40}+15=14.8$. Unbiased estimate for population variance is $s^2=\frac{1}{40-1}\left(\sum(t-15)^2-\frac{\left(\sum(t-15)\right)^2}{40}\right)=\frac{1}{39}\left(17-\frac{(-8)^2}{40}\right)$ $=\frac{77}{195}=0.3948717949$	Most students know the formula and calculate correctly.
(iii)	Let μ denotes the mean time of the worker in assembling the components for the electrical device. Test $H_0:\mu=15$ against $H_1:\mu\neq 15$ We conduct a 2-tail test at 5% level of significance Under H_0, we have $\bar{T}\sim N\left(15,\frac{77/195}{40}\right)$ approximately since $n=40$ is large. That is the test statistics is $Z=\frac{\bar{T}-15}{\sqrt{77/195}/\sqrt{40}}\sim N(0,\ 1)$ approximately. Using GC, we have p-value = 0.0441202641< 0.05  Thus, we reject H_0 and conclude that at 5% level of significance, there is sufficient evidence that the mean	Most students know the steps of Hypothesis testing

	time for each worker to complete the assembly process has changed and is not 15 minutes.	
(iv)	'5% level of significance' refers to the probability of 0.05 that the test concludes that the mean time for the workers to assemble the components of the electrical device has changed when in fact it has not.	Students need to answer in the context of the question and explain clearly. Quite a number of them said *it is the probability that mean has changed when in fact it did not*, which is incorrect. Whether the mean has changed or not we do not know and it is the probability of 0.05 act of wrongly concluding etc
(v)	It is <u>not necessary</u> for the production manager to assume that the time taken by a worker to completes the assembly process follows a normal distribution. Since the sample size is large, $\bar{X}$ is approximately normal.	Quite ok. But still quite a number of students said the *time* will follow normal which is incorrect.
(vi)	Using the available set of data from the 40 random chosen, we note that the p-value is 0.0441202641 for 2-tail test. Thus in the testing of $H_0 : \mu = 15$ against $H_1 : \mu < 15$, the p-value will then be half that of the previous value. So the p-value = 0.0220601321 > 0.02 Hence, we conclude that at 2% level of significance, there is insufficient evidence that the mean time for the worker to assemble the components of the electrical device has been reduced.	Quite ok for most students.
(vii)	For the 50 sets of data, $\sum_{n=1}^{50}(t_n - 15)$ $= \sum_{n=1}^{40}(t_n - 15) + \sum_{n=41}^{50}(t_n - 15)$ $= (-8) + (-1)$ $= -9$	This part proved to be challenging as students are not able to find the new unbiased estimator for variance.

$$\sum_{n=1}^{50}(t_n-15)^2$$
$$=\sum_{n=1}^{40}(t_n-15)^2+\sum_{n=41}^{50}(t_n-15)^2$$
$$=(17)+(0.26)$$
$$=17.26$$

Unbiased estimate for population mean is
$$\bar{t}=\frac{\sum(t-15)}{50}+15=\frac{-9}{50}+15=14.82.$$
Unbiased estimate for population variance is
$$s^2=\frac{1}{50-1}\left(\sum(t-15)^2-\frac{\left(\sum(t-15)\right)^2}{50}\right)$$
$$=\frac{1}{49}\left(17.26-\frac{(-9)^2}{50}\right)$$
$$=0.3191836735$$

Test $H_0:\mu=15$
against $H_1:\mu<15$

Test at $100\alpha\%$ level of significance

Since $n=50$ is large,
$$Z=\frac{\bar{T}-15}{\sqrt{\frac{0.3191836735}{50}}}\sim N(0,1)\text{ approximately.}$$

Using GC, we have
p-value = 0.0121334645

Since H_0 rejected, $\alpha>0.0121$