NAME: ______________________ (   )   CLASS: ______

FAIRFIELD METHODIST SCHOOL (SECONDARY)

PRELIMINARY EXAMINATION 2024
SECONDARY 4 EXPRESS

ADDITIONAL MATHEMATICS   4049/02

Paper 2

Date: 22 August 2024   **Duration: 2 hours 15 minutes**

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
The number of marks is given in brackets [ ] at the end of each question or part question.

If working is needed for any question it must be shown with the answer.
Omission of essential working will result in loss of marks.
The total of the marks for this paper is 90.

The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.
For π, use either your calculator value or 3.142.

For Examiner's Use

Table of Penalties		Question Number		
Presentation	☐ 1 ☐ 2			
Rounding off	☐ 1		Parent's/Guardian's Signature	/ 90

This question paper consists of 23 printed pages

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Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{r} a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

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Answer **all** the questions.

1 Show that the equation $2(e^x - 3) = e^{\frac{1}{2}x}$ has only one solution and find this value correct to 3 significant figures. [4]

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2 The equation of a curve is $y = ax^3 + b$, where a and b are constants. The equation of the normal to the curve at the point where $x = 1$ is $5y + 2x = 12$. Find the value of a and of b. [6]

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3 Given that $y = \frac{3\ln 2x}{x^2}$ for $x > 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{x^3}(1 - 2\ln 2x)$. [3]

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(b) Hence, find $\int \frac{\ln 2x}{x^3}\,\mathrm{d}x$. [3]

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4 Given that $\mathrm{f}(x) = x^2 - ax + 3$, where a is a constant,

(a) find the range of values of a for which $\mathrm{f}(x) > x - 1$, for all real values of x, [4]

(b) find the value(s) of a for which the line $y = a + 4$ is a tangent to the curve $y = \mathrm{f}(x)$. [3]

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5 **(a)** A and B are acute angles such that $\sin(A-B)=\frac{3}{8}$ and $\sin A\cos B=\frac{5}{8}$.

Without using a calculator, find the value of $\cos A\sin B$. [2]

(b) Express $2\sin 2\theta(\sec\theta-\tan\theta)$ as a quadratic expression in $\sin\theta$. [3]

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(c) Use your answer to part **(b)** to find, for $0 \le \theta \le 2\pi$, the exact solutions of the equation $2\sin 2\theta(\sec\theta - \tan\theta) + 3 = 0$. [3]

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6 A curve is such that $\frac{dy}{dx}=\frac{8}{x^2}-2$.

(i) Given that the curve passes through the point (1, 5), find the equation of the curve. [3]

(ii) Find the x–coordinates of the stationary points of the curve. [2]

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(iii) Obtain and expression for $\frac{d^2y}{dx^2}$ and hence, or otherwise, determine the nature of each stationary point. [3]

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7 **(a)** Write down the first three terms in the expansion, in descending powers of x, of $\left(x^2+\frac{m}{x}\right)^9$, where m is an integer. [2]

(b) Given that the coefficient of x^3 in the expansion of $\left(x^2+\frac{m}{x}\right)^9$ is -126,

(i) show that $m=-1$, [3]

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(ii) hence, find the term independent of x in the expansion of

$$\left(2-\frac{1}{x^3}\right)\left(x^2+\frac{m}{x}\right)^9.$$ [4]

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8 The table shows the population, P, in thousands, of a small town decreases with time, t years.

t	6	9	12	15	18	25	33
P	274	203	151	112	83	41	18

(a) Show your working clearly and draw a straight line graph of $\ln P$ against t on the grid provided. [3]

(b) Find the gradient of your straight line and hence express P in the form of Ae^{-kt}, where A and k are constants. giving your answers correct to the nearest hundred and to 1 decimal place respectively. [4]

(c) If this model for the population remains valid, find the number of years it will take for the population of the small town to drop below 100000. Give your answer correct to the nearest year. [2]

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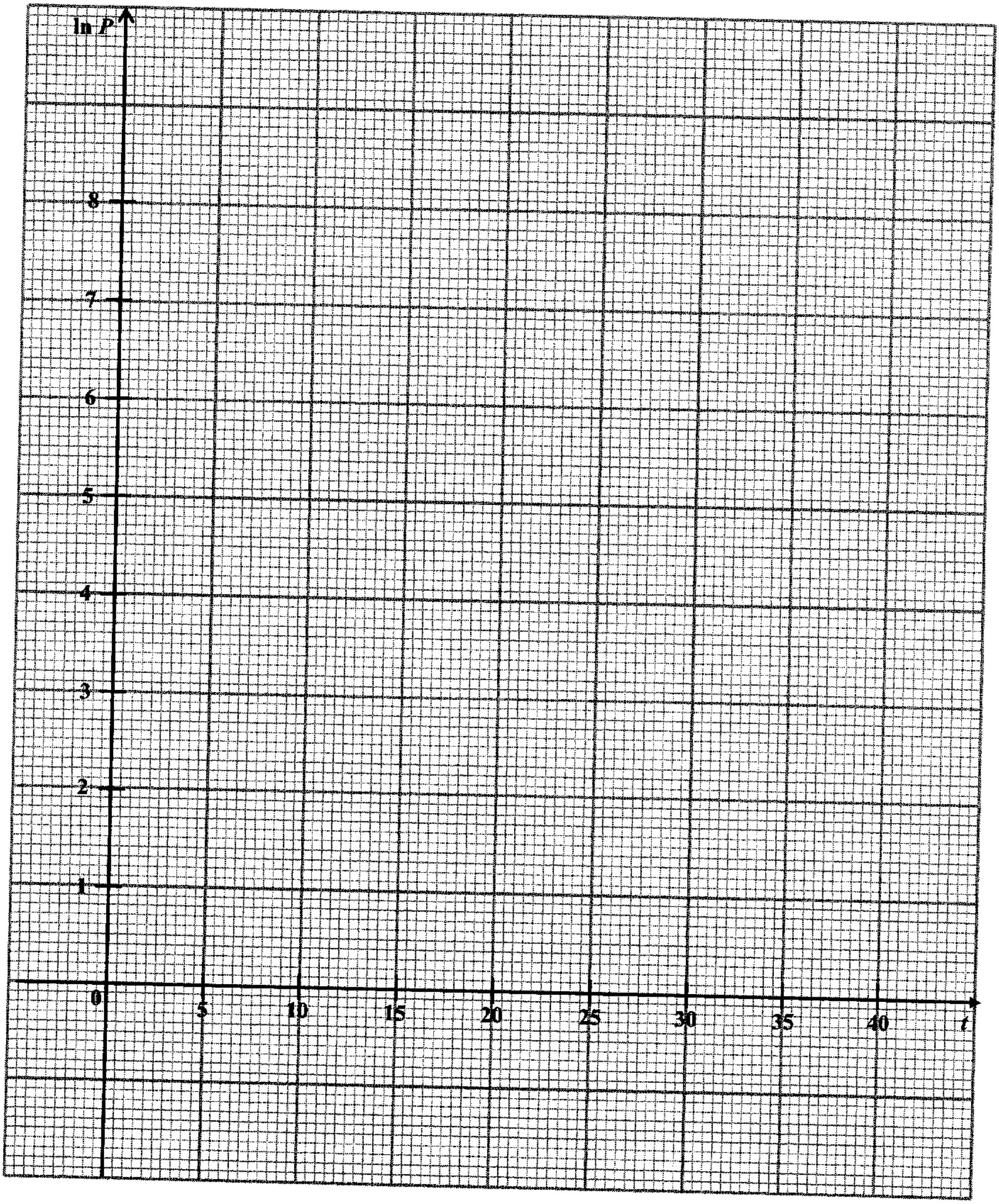
8 (a)



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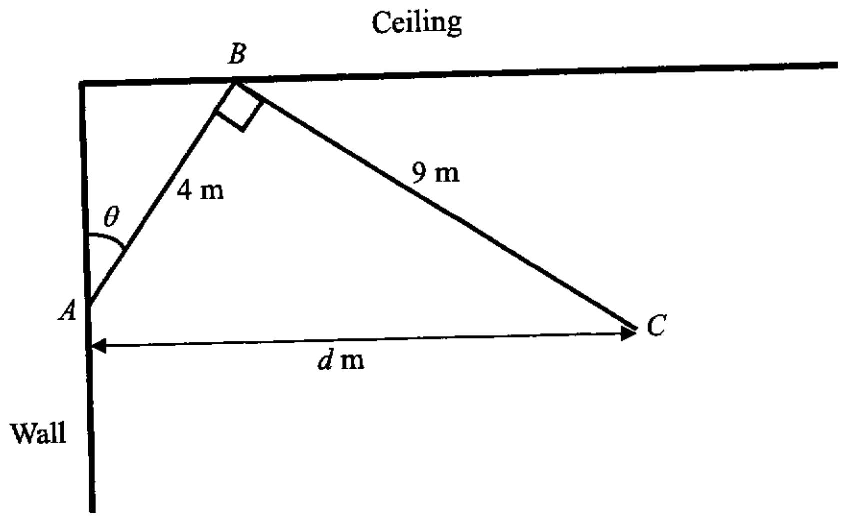
9 The diagram above shows two rods AB and BC of length 4 m and 9 m respectively and $\angle ABC = 90°$. Rollers are fixed at points A and B such that A is able to move along the wall and B is able to move along the celling. The horizontal distance of C from the vertical wall is d m.



(a) Show clearly that $d = 4\sin\theta + 9\cos\theta$. [2]

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(b) Express d in the form $R\cos(\theta - \alpha)$ where $R > 0$ and $0° < \alpha < 90°$. [4]

(c) Find the value of θ for which $d = 6$ m. [2]

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(d) Find the maximum value of d and the corresponding value of θ. [2]

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10 A particle moves in a straight line such that, t s after passing through a fixed point O, its displacement from O is s m. The velocity v ms^{-1} of the particle is such that $v = 6\cos 4t$.

(a) State the initial velocity of the particle. [1]

(b) Find the first value of t when the acceleration of the particle is equal to 8 ms^{-2}. [2]

(c) Find the displacement of the particle from O when $t = 4$. [3]

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(d) Find the total distance travelled by the particle for the first $\frac{3\pi}{8}$ seconds. [5]

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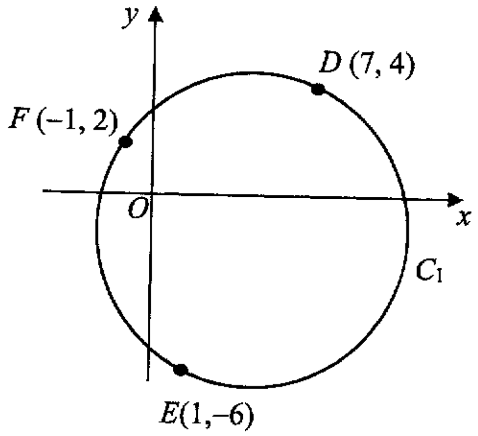
11 **The diagram below is not drawn to scale.**



In the diagram, D, E and F are points on the circle C_1.

(a) Show that DE is the diameter of the circle C_1 and hence find the centre of C_1. [5]

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(b) Find the equation of the circle C_1 in the form $x^2+y^2+px+qy+r=0$, where p, q and r are integers. [3]

(c) Given that the circle C_2 is a reflection of the circle C_1 in the line $x=-2$, find the equation of C_2. [2]

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(d) Explain why point (3, 4) lies within only one of the circles C_1 and C_2. [2]

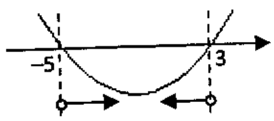
~ End of Paper ~

Sec 4 Add Math Preliminary Exam 2024 P2 Marking Scheme

Qn.	Solution	Marks	AO
1	$2(e^x - 3) = e^{\frac{1}{2}x}$ Let $y = e^{\frac{1}{2}x}$, $2(y^2 - 3) = y$ $2y^2 - y - 6 = 0$ $(2y+3)(y-2) = 0$ $y = -\frac{3}{2}$ or $y = 2$ $e^{\frac{1}{2}x} = -\frac{3}{2}$ (rej. as -ve) $e^{\frac{1}{2}x} = 2$ $\frac{1}{2}x = \ln 2$ $x = 2\ln 2$ $= 1.39$ (3 s.f) Therefore, the equation has only one solution where x=1.39	M1 (substitution) M1 (factorization) A1 (reject $-\frac{3}{2}$) (Did not award marks for students who squared both sides and could not justify why they rejected one answer when they ended up with 2 answers) AG1	3

2	$y = ax^3 + b$ $\frac{dy}{dx} = 3ax^2$ $5y + 2x = 12$ $5y = -2x + 12$ $y = -\frac{2}{5}x + \frac{12}{5}$ Grad of normal = $-\frac{2}{5}$ Grad of tangent = $\frac{5}{2}$ $\therefore \frac{dy}{dx} = \frac{5}{2}$ At $x = 1$, $y = -\frac{2}{5}(1) + \frac{12}{5}$ $y = 2$ At $x = 1, y = 2$ $2 = a(1)^3 + b$ $a + b = 2$ ------ (1) At $x = 1, \frac{dy}{dx} = \frac{5}{2}$ $\frac{5}{2} = 3a(1)^2$ ------ (2) $a = \frac{5}{6}$ Substituting $a = \frac{5}{6}$ into (1) $\frac{5}{6} + b = 2$ $b = \frac{7}{6}$	B1 (find $\frac{dy}{dx}$) B1 (grad of tangent) M1 (find $y = 2$) M1 (for either (1) or (2)) A1 (for a) A1 (for b)	2

3a	$y=\frac{3\ln(2x)}{x^2}$ $\frac{dy}{dx}=\frac{x^2\left(\frac{3(2)}{2x}\right)-6x\ln 2x}{\left(x^2\right)^2}$ $=\frac{3x-6x\ln 2x}{x^4}$ $=\frac{3}{x^3}(1-2\ln 2x)$	M1 (Apply quotient rule) M1 (Able to diff $\ln 2x=\frac{2}{2x}=\frac{1}{x}$.) A1	1
3b	$\int\left(\frac{3}{x^3}-\frac{6}{x^3}\ln 2x\right)\mathrm{d}x=\frac{3\ln 2x}{x^2}+C$ $\int\left(\frac{1}{x^3}-\frac{2}{x^3}\ln 2x\right)\mathrm{d}x=\frac{\ln 2x}{x^2}+C_1$ $\int\frac{2\ln 2x}{x^3}\mathrm{d}x=-\frac{\ln 2x}{x^2}-\frac{1}{2x^2}+C_2$ $\int\frac{\ln 2x}{x^3}\mathrm{d}x=-\frac{1}{2}\left(\frac{\ln 2x}{x^2}+\frac{1}{2x^2}\right)+C_3$	M1 (reverse differentiation – must include + C) M1 (integrate $\frac{1}{x^3}$) A1(must include + C) (Whole question will only deduct once if they did not put + C)	2

4a	$f(x) > x-1$ $x^2 - ax + 3 > x - 1$ $x^2 - (a+1)x + 4 > 0$ Since it is always positive for all real values of x, the graph of the curve $y = x^2 - (a+1)x + 4$ lies entirely above the x-axis $\Rightarrow$ the equation $x^2 - (a+1)x + 4 = 0$ has no real roots Discriminant, $D < 0$ $(a+1)^2 - 4(1)(4) < 0$ $(a+1)^2 - 4^2 < 0$ $(a+5)(a-3) < 0$  $\therefore -5 < a < 3$	M1 (form inequality) (students who equated both eqns together will not get M1 unless they explain that there are no real roots and lead to $D<0$) M1 (Discriminant less than zero) M1 (Factorization) A1	2
4b	$\begin{cases} y = f(x) \\ y = a+4 \end{cases}$ $x^2 - ax + 3 = a + 4$ $x^2 - ax - a - 1 = 0$ Since the line $y = a+4$ is a tangent to the curve, this equation has equal real roots Discriminant, $D = 0$ $(-a)^2 - 4(1)(-a-1) = 0$ $a^2 + 4a + 4 = 0$ $(a+2)(a+2) = 0$ $\therefore a = -2$	M1 (equate equations together) M1 ($D = 0$) A1	1
5a	$\sin(A-B) = \sin A \cos B - \cos A \sin B$ $\cos A \sin B = \sin A \cos B - \sin(A-B)$ $= \frac{5}{8} - \frac{3}{8}$ $= \frac{1}{4}$	M1 (make $\cos A \sin B$ the subject) A1	1

5b	$2\sin 2\theta(\sec\theta-\tan\theta)$ $=2(2\sin\theta\cos\theta)\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)$ $=2(2\sin\theta\cos\theta)\left(\frac{1-\sin\theta}{\cos\theta}\right)$ $=4\sin\theta-4\sin^2\theta$	M1 (double angle formula) M1 (bring $\cos\theta$ under same denominator) A1	2
5c	$2\sin 2\theta(\sec\theta-\tan\theta)+3=0$ $4\sin\theta-4\sin^2\theta+3=0$ $4\sin^2\theta-4\sin\theta-3=0$ $(2\sin\theta+1)(2\sin\theta-3)=0$ $\sin\theta=-\frac{1}{2}$ or $\sin\theta=\frac{3}{2}$ (rejected) $\alpha=\frac{\pi}{6}$ $\theta=\pi+\frac{\pi}{6},2\pi-\frac{\pi}{6}$ $\theta=\frac{7\pi}{6},\frac{11\pi}{6}$	M1 (factorize) M1 (show both and must reject one) A1 (must be in terms of π as qn asked for exact solns)	1
6(i)	$y=\int\left(\frac{8}{x^2}-2\right)dx$ $y=-\frac{8}{x}-2x+c$ At $x=1, y=5$ $5=-\frac{8}{1}-2(1)+c$ $c=15$ Equation of curve: $y=-\frac{8}{x}-2x+15$	M1 (with $+c$) M1 (Sub in values) A1	1
6(ii)	Let $\frac{dy}{dx}=0$ $\frac{8}{x^2}-2=0$ $\frac{8}{x^2}=2$ $x^2=4$ $x=2$ or $x=-2$	M1 A1 (both answers)	1

6(iii)	$\frac{d^2y}{dx^2}=-\frac{16}{x^3}$ At $x=2$, $\frac{d^2y}{dx^2}=-\frac{16}{2^3}=-2<0$ $\therefore$ Maximum point at $x=2$ At $x=-2$, $\frac{d^2y}{dx^2}=-\frac{16}{(-2)^3}=-\frac{16}{-8}=2>0$ $\therefore$ Minimum point at $x=-2$	M1 (2^{nd} derivative) A1 A1 (for students who got part (ii) wrong, maximum mark is 1M if M1 shown)	1
7a	$\left(x^2+\frac{m}{x}\right)^9=x^{18}+\binom{9}{1}\left(x^2\right)^8\left(\frac{m}{x}\right)^1+\binom{9}{2}\left(x^2\right)^7\left(\frac{m}{x}\right)^2+....$ $=x^{18}+9\,mx^{15}+36m^2x^{12}+....$	B2 (3 terms all correct) B1(2 terms correct)	1
7b(i)	For $\left(x^2+\frac{m}{x}\right)^9$, general term is $(r+1)^{th}$ term $=\binom{9}{r}\left(x^2\right)^{9-r}(m)^r\left(x^{-1}\right)^r$ $=\binom{9}{r}(m)^r\left(x^{18-2r}\right)\left(x^{-r}\right)$ $=\binom{9}{r}(m)^r\left(x^{18-3r}\right)$ $18-3r=3$ $r=5$ $\therefore -126=\binom{9}{5}m^5$ $-126=126m^5$ $m^5=-1$ $m=-1$	M1 (general term) A1 ($r=5$) AG1	3

7b(ii)

For $\left(x^2+\frac{m}{x}\right)^9$, the term independent of x:

$18-3r=0$
$r=6$

$\therefore \binom{9}{6}m^6 = 84(-1)^6 = 84$

For $\left(2-\frac{1}{x^3}\right)\left(x^2+\frac{a}{x}\right)^9$, the term independent of x:

$\left(2-\frac{1}{x^3}\right)\left(\ldots-126x^3+84+\ldots\right)^9$

$=2\times 84+126$

$=294$

M1

A1

M1

A1

Marks: 1

8a

t (years)	6	9	12	15	18	25	33
P	274	203	151	112	83	41	18
$\ln P$	5.61	5.31	5.02	4.72	4.42	3.71	2.89

P2 – Plot points accurately.

L1 – Plot straight line graph

(See graph attached.)

Marks: 1

8b

$P = Ae^{-kt}$

$\ln P = \ln Ae^{-kt}$
$\ln P = \ln A + \ln e^{-kt}$
$\ln P = \ln A - kt$
$\ln P = -kt + \ln A$

M1 (product law or if evidence shown in transformation from eqn of graph to $P = Ae^{-kt}$)

$\text{Grad} = \frac{5.61-4.72}{6-15}$
$= -0.0989$ (3 s.f)
$-k = -0.0989$
$k = 0.1$(1 d.p)
$\ln A = 6.2$
$A = 492.75$
$A = 500$ (nearest 100)

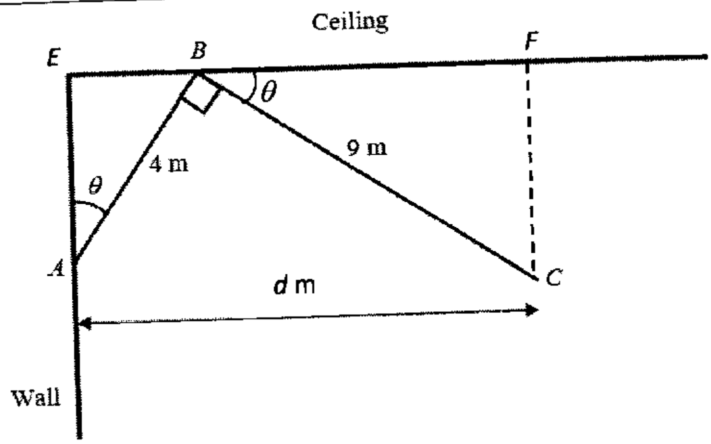
M1 (gradient)

A1

$\because P = 500e^{-0.1t}$

B1
(If students did not use the gradient of line to solve for k and A, maximum mark is 1M as question mentioned hence.)

Marks: 2

8c	$\ln 100 = 4.6$ When $\ln P = 4.6$ $t = 16$ (nearest year)	M1 A1	1
9a	 $EB = 4\sin\theta$ $BF = 9\cos\theta$ $d = 4\sin\theta + 9\cos\theta$	} M1 AG 1	3
9b	$R = \sqrt{81+16}$ $R = \sqrt{97}$ $9\cos\theta + 4\sin\theta = R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$ $9 = R\cos\alpha$ $4 = R\sin\alpha$ $\tan\alpha = \frac{4}{9}$ $\alpha = 23.962°(3d.p)$ $\alpha = 24.0°(1d.p)$ $9\cos\theta + 4\sin\theta = \sqrt{97}\cos(\theta - 24.0°)$	M1 (Find R) M1 (No M1 given if student do not show this) M1 (Find α) A1	1
9c	$\sqrt{97}\cos(\theta - 23.96°) = 6$ $\cos(\theta - 23.96°) = \frac{6}{\sqrt{97}}$ $\theta = 52.467° + 23.96°$ $\theta = 76.427° \approx 76.4°$	M1 A1 (no A1 if 76.5)	1

9d	Maximum value of $d = \sqrt{97}$ $\cos(\theta - 24.0°) = 1$ $\theta = 24.0$ Maximum value of $d = \sqrt{97}$ and occurs when $\theta = 24.0°$	B1 B1	1
10a	$v = 6\cos 4t$ When $t = 0, v = 6$ Initial velocity of the particle is 6m/s.	B1	1
10b	$a = \frac{dv}{dt} = -24\sin 4t$ $-24\sin 4t = 8$ $\sin 4t = -\frac{1}{3}$ $4t = 3.4814$ $t = 0.870$ (3 s.f)	M1 ($\frac{dv}{dt}$) A1	2
10c	$s = \int 6\cos 4t\ dt$ $s = \frac{6}{4}\sin 4t + c$ $s = \frac{3}{2}\sin 4t + c$ When $t = 0$, $s = 0$, $c = 0$ $s = \frac{3}{2}\sin 4t$ When $t = 4$, $s = \frac{3}{2}\sin 16$ $s = -0.432$ Displacement = -0.432m (3 sf)	M1 (integration with + c) A1 (conclude c=0) (students who used definite integral must indicate when t = 0, s=0) B1	1
10d	At instantaneous rest, $v = 0$ $6\cos 4t = 0$ $\cos 4t = 0$ $4t = \frac{\pi}{2}, \frac{3\pi}{2}$ $t = \frac{\pi}{8}, \frac{3\pi}{8}$	M1 (v= 0) M1 (values of t)	2

	When $t=\frac{\pi}{8}$, $s=\frac{3}{2}\sin\frac{\pi}{2}=1.5\text{m}$ When $t=\frac{3\pi}{8}$, $s=\frac{3}{2}\sin\frac{3\pi}{2}=-1.5\text{m}$ Total distance travelled $=(1.5\times 2)+1.5$ $=4.5\text{m}$	M1 M1 A1	
11a	**<u>Solution 1</u>** Grad EF $=\frac{2-(-6)}{-1-1}=-4$ Grad DF $=\frac{4-2}{7-(-1)}=\frac{1}{4}$ Since Grad $DF\times$ Grad $EF=-4\times\frac{1}{4}=-1$ $\therefore DF\perp EF$ $\angle DFE=90°$ (Right angle in semi-circle) $\therefore DE$ is the diameter of C_1 Since PQ is the diameter of C_1 Centre of $C_1=\left(\frac{1+7}{2},\frac{-6+4}{2}\right)$ $=(4,-1)$	M1 (Find grad) B1 (Show – 1) B1(Conclude 90°) AG 1 (State reason angle in semi-circle) B1	3
11a	**<u>Solution 2</u>** $DF=\sqrt{(7+1)^2+(4-2)^2}=\sqrt{68}$ $EF=\sqrt{(1+1)^2+(-6-2)^2}=\sqrt{68}$ $DE=\sqrt{(7-1)^2+(4+6)^2}=\sqrt{136}$ $DF^2+EF^2=\left(\sqrt{68}\right)^2+\left(\sqrt{68}\right)^2=136$ $DE^2=\left(\sqrt{136}\right)^2=136$ $\because DF^2+EF^2=DE^2$ By converse of Pythagoras theorem, triangle DFE is a right angled triangle. Therefore $\angle DFE=90°$.	M1 (Find distance) B1 (Show this statement) B1(Conclude 90° - must state Pythagoras thm)	3

	Since $\angle DFE = 90°$ By converse of right angle in semi-circle $\therefore DE$ is the diameter of C_1 Since PQ is the diameter of C_1 Centre of $C_1 = \left(\frac{1+7}{2}, \frac{-6+4}{2}\right)$ $= (4,-1)$	AG 1 (State reason angle in semi-circle) B1	
11b	Radius of $C_1 = \sqrt{(4-1)^2+(-1-(-6))^2} = \sqrt{34}$ Equation of C_1: $(x-4)^2+(y+1)^2=34$ $x^2+y^2-8x+2y-17=0$	B1 M1 A1	1
11c	Centre of $C_2 = (-8,-1)$ Radius $= \sqrt{34}$ Equation of C_2: $(x+8)^2+(y+1)^2=34$	B1 (centre) B1	2
11d	Distance of E from centre of C_1 $=\sqrt{(3-4)^2+(4+1)^2}$ $= \sqrt{26}$ $<$ radius of C_1 Distance of E from centre of C_2 $=\sqrt{(3+8)^2+(4+1)^2}$ $= \sqrt{146}$ $>$ radius of C_2 $\therefore (3, 4)$ lies within C_1 only	M1 A1	2